

AN ARTIN-REES THEOREM AND APPLICATIONS TO ZERO CYCLES

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ABSTRACT. For the smooth normalization $f : \overline{X} \rightarrow X$ of a singular variety X over a field k of characteristic zero, we show that for any conducting subscheme Y for the normalization, and for any $i \in \mathbb{Z}$, the natural map $K_i(X, \overline{X}, nY) \rightarrow K_i(X, \overline{X}, Y)$ is zero for all sufficiently large n .

As an application, we prove a formula for the Chow group of zero cycles on a quasi-projective variety X over k with Cohen-Macaulay isolated singularities, in terms of an inverse limit of relative Chow groups of a desingularization \tilde{X} relative to the multiples of the exceptional divisor.

We use this formula to verify a conjecture of Srinivas about the Chow group of zero cycles on the affine cone over a smooth projective variety which is arithmetically Cohen-Macaulay.

1. INTRODUCTION

It has been shown over the years that the relative and the double relative K -groups play a very important role in the study of algebraic K -theory and algebraic cycles on singular varieties. These are particularly very useful tools in comparing the K -groups (or Chow groups) of a singular scheme and its normalization. The double relative K -groups are in general very difficult to compute. Our aim in this paper is two fold. First we prove an Artin-Rees type theorem for the double relative K -groups, and then give some very important applications of this result to convince the reader why such Artin-Rees type results are indeed desirable if one wants to study algebraic cycles on singular varieties.

In this paper, a *variety* will mean a reduced, connected and separated scheme of finite type over a field k of characteristic zero. All the underlying fields in this paper will be of characteristic zero. Let X be singular variety of dimension d over a field k and let $f : \overline{X} \rightarrow X$ be the normalization morphism. For any closed subscheme Y of X , the relative K -groups $K_*(X, Y)$ are defined as the stable homotopy groups of the homotopy fiber $K(X, Y)$ of the map $K(X) \rightarrow K(Y)$ of non-connective spectra as defined in [27]. If $Y \hookrightarrow X$ is a conducting subscheme for the normalization, we put $\overline{Y} = Y \times_X \overline{X}$. We define the double relative K -groups $K_*(X, \overline{X}, Y)$ as the stable homotopy groups of the homotopy fiber $K(X, \overline{X}, Y)$ of the induced natural map $K(X, Y) \rightarrow K(\overline{X}, \overline{Y})$ of non-connective spectra. As shown in [27], the underlying spectra to define the relative and double relative K -groups are non-connective, and hence the groups $K_i(X, \overline{X}, Y)$ may be non-zero

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even when i is negative. For a conducting subscheme Y , we let \mathcal{I} be the sheaf of ideals on X defining Y . We denote by nY (for $n \geq 1$), the conducting subscheme of X for the normalization that is defined by the sheaf of ideals \mathcal{I}^n . The functoriality of the double relative groups defines the natural maps $K_*(X, \overline{X}, (n+1)Y) \rightarrow K_*(X, \overline{X}, nY)$ for all $n \geq 1$.

If X is affine, it is classically known that $K_i(X, \overline{X}, Y) = 0$ for $i \leq 0$. Furthermore, it follows from [9] (Theorem 3.6) and a result of Cortinas ([8], Corollary 0.2) that $K_1(X, \overline{X}, Y) \cong \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes \Omega_{\overline{Y}/Y}$. This shows that for X affine, the map $K_i(X, \overline{X}, 2Y) \rightarrow K_i(X, \overline{X}, Y)$ is zero for $i \leq 1$. This has already been very useful in the study of zero cycles on singular schemes. We refer the reader to [16], [14], [3] among several others for some applications of such results to zero cycles on surfaces and threefolds. One of the main difficulties in advancing the study of K -theory and algebraic cycles to the higher dimensional singular varieties has been the need for the following general Artin-Rees type result.

Theorem 1.1. *Let X be a quasi-projective variety of dimension d over a field k and let $f : \overline{X} \rightarrow X$ be the normalization of X . Assume that \overline{X} is smooth. Then for any conducting subscheme $Y \hookrightarrow X$ for the normalization, and for any $i \in \mathbb{Z}$, the natural map $K_i(X, \overline{X}, nY) \rightarrow K_i(X, \overline{X}, Y)$ is zero for all sufficiently large n .*

As already said before, one of the motivations for proving the above theorem is its applications in computing the K -groups and the Chow groups of algebraic cycles on singular varieties. These Chow groups are usually very difficult to compute, even for varieties with isolated singularities. In this paper, we use the above theorem to prove the following formula for the Chow group of zero cycles on a quasi-projective variety X with Cohen-Macaulay isolated singularities, in terms of an inverse limit of the relative Chow groups of a desingularization \tilde{X} relative to the multiples of the exceptional divisor. Such a formula for the normal surface singularities was conjectured by Bloch and Srinivas ([24]) and proved by Srinivas and the author in [16] (Theorem 1.1). This formula was later verified for the threefolds with Cohen-Macaulay isolated singularities in [14] (Theorem 1.1).

Before we state our next theorem, we very briefly recall the definition of the Chow group of zero cycles on singular varieties and some other related notions. Let X be a quasi-projective singular variety of dimension d over a field k . The cohomological *Chow group of zero cycles* $CH^d(X)$ was defined by Levine and Weibel in [19] as the free abelian group on the smooth closed points of X , modulo the zero cycles given as the sum of divisors of suitable rational functions on certain *Cartier curves* on X . These curves can be assumed to be irreducible and disjoint from the singular locus of X , if it is normal. We refer the reader to [12] for the full definition of $CH^d(X)$ and some of its important properties.

Let $F^d K_0(X)$ denote the subgroup of the Grothendieck group of vector bundles $K_0(X)$, generated by the classes of smooth codimension d closed points on X . There is a natural surjective map $CH^d(X) \rightarrow F^d K_0(X)$. This map is known (cf. [17], Corollary 2.6 and Theorem 3.2) to be an isomorphism, if the underlying field k is algebraically closed and X is either an affine or a projective variety over k with only normal isolated singularities. This isomorphism will be used throughout this paper. For any closed subscheme Z of X , let $F^d K_0(X, Z)$ be the subgroup

of $K_0(X, Z)$ (defined above) generated by the classes of smooth closed points of $X - Z$ (cf. [7]). We shall also call $F^d K_0(X, Z)$ the *relative Chow group of zero cycles of X relative to Z* .

Now let X be an irreducible quasi-projective variety of dimension d with isolated singularities over a field k , and let $p : \tilde{X} \rightarrow X$ be a resolution of singularities of X . Let E denote the reduced exceptional divisor on \tilde{X} and let nE denote the n th infinitesimal thickening of E . From the above definitions and from Lemma 7.1 below (see also [14], Theorem 1.1), one has the following commutative diagram for each $n > 1$ with all arrows surjective.

$$(1.1) \quad \begin{array}{ccccc} & & F^d K_0(X) & & \\ & \swarrow & \downarrow & \searrow p^* & \\ F^d K_0(\tilde{X}, nE) & \longrightarrow & F^d K_0(\tilde{X}, (n-1)E) & \longrightarrow & F^d K_0(\tilde{X}) \end{array}$$

We also recall from [14] that a resolution of singularities $p : \tilde{X} \rightarrow X$ of X is called a *good resolution of singularities* if p is obtained as a blow up of X along a closed subscheme Z whose support is the set of singular points of X . Such a resolution of singularities always exists. In fact, it is known (cf. [14], Lemma 2.5) using Hironaka's theory that every resolution of singularities is dominated by a good resolution of singularities. Hence in the study of the Chow group of zero cycles on X , it suffices to consider only the good resolutions of singularities since $CH^d(\tilde{X})$ is a birational invariant of a smooth projective variety \tilde{X} of dimension d . We now state our next result.

Theorem 1.2. *Let X be a quasi-projective variety of dimension $d \geq 2$ over a field k . Let $p : \tilde{X} \rightarrow X$ be a good resolution of singularities of X with the reduced exceptional divisor E on \tilde{X} . Assume that X has only Cohen-Macaulay isolated singularities. Then for all sufficiently large n , the maps*

(i) $F^d K_0(\tilde{X}, nE) \rightarrow F^d K_0(\tilde{X}, (n-1)E)$ and

(ii) $F^d K_0(X) \rightarrow F^d K_0(\tilde{X}, nE)$

are isomorphisms. In particular, if $k = \bar{k}$ and X is either affine or projective, then

$$CH^d(X) \cong \varprojlim_n F^d K_0(\tilde{X}, nE).$$

Since rational singularities are Cohen-Macaulay, we immediately get

Corollary 1.3. *If X is a quasi-projective variety with only rational and isolated singularities, then one has the above formula for $CH^d(X)$.*

It turns out that the relative Chow groups (as defined above) of the resolution of singularities, relative to the multiples of the exceptional divisor are often computable. Such computations have been carried out before to compute the Chow groups of zero cycles on normal surfaces (cf. [16]) and certain threefolds (cf. [14]). We now proceed to carry these computations further in all dimensions and then use Theorem 1.2 to compute the Chow group of zero cycles on certain classes of singular varieties. These results in particular give many new examples of the validity

of the singular analogue of the well known conjecture Bloch and its generalization in higher dimensions.

Our result in this direction is inspired by the following conjecture of Srinivas ([25], Section 3), which in itself was probably motivated by the affine analogue of a conjecture of Bloch about the Chow groups of zero cycles on smooth projective varieties.

Conjecture 1.4 (Srinivas). *Let $Y \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ be a smooth and projectively normal variety over \mathbb{C} of dimension d and let $X = C(Y)$ be the affine cone over Y . Then $CH^{d+1}(X) = 0$ if and only if $H^d(Y, \mathcal{O}_Y(1)) = 0$.*

The ‘only if’ part of this conjecture was proved by Srinivas himself in [25] (Corollary 2). So the ‘if’ part remains open at present. This conjecture was verified in [16] (Corollary 1.4) when Y is a curve. This was also verified by Consani ([6], Proposition 3.1) in a very special case when Y is a hypersurface in \mathbb{P}^3 . The following result verifies this conjecture when the affine cone X is Cohen-Macaulay. In particular, Srinivas’ conjecture is verified when Y is a complete intersection in \mathbb{P}^N .

Theorem 1.5. *Let k be an algebraically closed field of characteristic zero and let $Y \hookrightarrow \mathbb{P}_k^N$ be a smooth projective variety of dimension d . Let $X = C(Y)$ be the affine cone over Y . Assume that X is Cohen-Macaulay. Then $CH^{d+1}(X) = 0$ if $H^d(Y, \mathcal{O}_Y(1)) = 0$.*

Moreover, if k is a universal domain and if \overline{X} is the projective cone over Y , then the following are equivalent.

- (i) $H^d(Y, \mathcal{O}_Y(1)) = 0$
- (ii) $H^{d+1}(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$
- (iii) $H^{d+1}(\overline{X}, \Omega_{\overline{X}/k}^i) \xrightarrow{\cong} H^{d+1}(W, \Omega_{W/k}^i)$ for all $i \geq 0$, where W is a resolution of singularities of \overline{X} .
- (iv) $CH^{d+1}(X) = 0$
- (v) $CH^{d+1}(\overline{X}) \cong CH^d(Y)$.

If $Y \hookrightarrow \mathbb{P}_k^N$ is a hypersurface of degree d , then there is an isomorphism $H^{N-1}(Y, \mathcal{O}_Y(1)) \cong H^0(\mathbb{P}_k^N, \mathcal{O}_{\mathbb{P}^N}(d - N - 2))$ and we obtain the following immediate consequence of Theorem 1.5.

Corollary 1.6. *Let $Y \hookrightarrow \mathbb{P}_k^N$ be a smooth hypersurface of degree d and let $X = C(Y)$ be the affine cone over Y . Then $CH^N(X) = 0$ if $d \leq N + 1$. The converse also holds if k is a universal domain.*

The following application of Theorem 1.5 to the projective modules on singular affine algebras follows at once from Murthy’s result ([21], Corollary 3.9).

Corollary 1.7. *Let $Y \hookrightarrow \mathbb{P}_k^N$ be as in Theorem 1.5 and let A be its homogeneous coordinate ring. If $H^d(Y, \mathcal{O}_Y(1)) = 0$, then every projective module over A of rank at least d has a unimodular element.*

If X is a normal projective surface with a resolution of singularities $p : \tilde{X} \rightarrow X$, then it was shown in [16] (Theorem 1.3) that $CH^2(X) \cong CH^2(\tilde{X})$ iff $H^2(X, \mathcal{O}_X) \cong$

$H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. If X has higher dimension, it is not expected that the isomorphism of the top cohomology of the structure sheaves is the sufficient condition to conclude the isomorphism of the Chow groups of zero cycles. However, Theorem 1.5 suggests that this might still be true in the case of isolated singularities.

As can be seen in the above results, our applications of Theorem 1.1 has been restricted to computing some pieces of the group K_0 of singular varieties. On the other hand, the conclusion of this theorem applies also to higher K_i 's. We hope that this result will be a significant tool to study K_1 of singular varieties for which hardly anything is known.

We conclude this section with a brief outline of this paper. Our strategy of proving Theorem 1.1 is to use the Brown-Gersten spectral sequence of [27] and the Cortinas' proof of KABI-conjecture in [8] to reduce the problem to proving similar results for the Hochschild and cyclic homology. Since these homology groups have canonical decompositions in terms of André-Quillen homology, we first prove an Artin-Rees theorem for these homology groups. We give an overview of the André-Quillen homology and then generalize a result of Quillen ([23], Theorem 6.15) in the next section. Section 3 contains a proof of our Artin-Rees type result for the André-Quillen homology. In Section 4, we deal with proving some refinements of this result for the special case of conducting ideals for the smooth normalization of the essentially of finite type k -algebras, where k is a field. Section 5 generalizes these results for the André-Quillen homology over any base field which is contained in k . We then use these results in Section 6 to prove the Artin-Rees theorem for the double relative Hochschild and cyclic homology, and then give the proof of Theorem 1.1. Section 7 contains the proof of Theorem 1.2. In Section 8, we relate the Chow group of zero cycles with certain cohomology of Milnor K -sheaves and then prove some general results about these cohomology groups, which are then used in the final section to compute the Chow group of zero cycles on the affine cones.

2. ANDRÉ-QUILLEN HOMOLOGY

All the rings in this section will be assumed to be commutative k -algebras, where k is a given field of characteristic zero. Our aim in this section is to give an overview of André-Quillen homology and related concepts. This homology theory of algebras will be our main tool to prove Theorem 1.1. We also prove here a generalization of an Artin-Rees type theorem of Quillen [23], Theorem 6.15) for the André-Quillen homology of algebras which are finite over the base ring. As Quillen shows (see also [1]), such a result itself has many interesting consequences for the homology of commutative algebras. Apart from the above cited works of André and Quillen, our other basic reference for this material including the Hochschild and cyclic homology, is [20].

Let A be a commutative ring which is essentially of finite type over the field k , and let B be an A -algebra. A simplicial A -algebra will mean a simplicial object in the category of A -algebras. Let P_* be a simplicial A -algebra. We say that P_* is B -augmented if the natural map $A \rightarrow B$ factors through $A \rightarrow P_* \rightarrow B$, where any A -algebra B is naturally considered a simplicial A -algebra with all the face and degeneracy maps taken as identity map of B . The homotopy groups of a simplicial A -algebra P_* is defined as the homotopy groups of the simplicial set P_* , which is same as the homotopy groups of the simplicial A -module P_* . The Dold-Kan correspondence implies that these homotopy groups are same as the

homology groups of the corresponding chain complex (which we also denote by P_*) of A -modules. Using this equivalence between simplicial A -modules and chain complexes of A -modules, we shall often write the homotopy groups $\pi_i(M_*)$ of a simplicial A -module M_* homologically as $H_i(M_*)$ without any ado. We say that P_* is a *free A -algebra* if each P_i ($i \geq 0$) is a symmetric algebra over a free A -module.

Definition 2.1. A free simplicial A -algebra P_* is called a *simplicial resolution* of an A -algebra B if P_* is B -augmented such that the natural map $H_i(P_*) \rightarrow H_i(B)$ is an isomorphism for all i .

It is known (cf. [20], Lemma 3.5.2) that any A -algebra B admits a free simplicial resolution and any two such resolutions are homotopy equivalent.

Before we define the cotangent modules, we recall (cf. [20], 1.6.8) that for two simplicial B modules M_* and N_* , their tensor and wedge products are defined degree-wise, i.e.,

$$(M_* \otimes_B N_*)_i = M_i \otimes_B N_i \text{ and } (\wedge_B^r M_*)_i = \wedge_B^r M_i \text{ for } r, i \geq 0.$$

The face and degeneracy maps of the tensor (or wedge) product are degree-wise tensor (or wedge) product of the corresponding maps. Since we are in characteristic zero, the following lemma relating tensor and exterior powers of a simplicial B -module is elementary.

Lemma 2.2. *For any simplicial B -module M_* and for $r \geq 0$, $\wedge_B^r M_*$ is canonically a retract of $\otimes_B^r M_*$. In particular, $H_i(\wedge_B^r M_*)$ is a canonical direct summand of $H_i(\otimes_B^r M_*)$ for all $i \geq 0$.*

Proof. This is well known and we only give a very brief sketch. Since the tensor and exterior powers are defined degree-wise, it suffices to prove the lemma for a B -module M . One defines a B -linear map

$$\otimes^r M \xrightarrow{\text{alt}} \otimes^r M$$

as

$$\text{alt}(a_1 \otimes \cdots \otimes a_r) = 1/r! \sum_{\sigma \in S_r} \text{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}.$$

It is easy to check that ‘alt’ is a projector and is natural with respect to map of B -modules. Moreover, if $\widetilde{\wedge}^r M$ denotes the image of this map, then the composite $\widetilde{\wedge}^r M \rightarrow \otimes^r M \rightarrow \wedge^r M$ is a canonical isomorphism. \square

For an A -algebra B , we define the *cotangent module* of B to be the simplicial B -module $\mathbb{L}_{B/A}$ given by

$$(2.1) \quad (\mathbb{L}_{B/A})_i = \Omega_{P_i/A}^1 \otimes_{P_i} B,$$

where P_* is any free simplicial resolution of B and the face and degeneracy maps of $\mathbb{L}_{B/A}$ are induced by those of P_* . The homotopy equivalence of different simplicial resolutions of B implies that $\mathbb{L}_{B/A}$ is a well-defined object in the homotopy category of simplicial B -modules. The *André-Quillen homology* of B with coefficients in a B -module M is defined as

$$D_q(B/A, M) = H_q(\mathbb{L}_{B/A} \otimes_B M), \quad q \geq 0.$$

One defines the *higher André-Quillen homology* of B with coefficients in a B -module M as

$$D_q^{(r)}(B/A, M) = H_q \left(\mathbb{L}_{B/A}^{(r)} \otimes_B M \right) \text{ for } r, q \geq 0,$$

where

$$\mathbb{L}_{B/A}^{(r)} = \wedge_B^r (\mathbb{L}_{B/A}) \text{ for } r \geq 0.$$

When M is same as B , we write $D_q^{(r)}(B/A, M)$ simply as $D_q^{(r)}(B/A)$. It is easy to see from these definitions that $D_0^{(0)}(B/A, M) = M$ and $D_{\geq 1}^{(0)}(B/A, M) = 0$. It is also known (cf. [20], Theorem 3.5.8, Theorem 4.5.12) that for any $r \geq 0$, there is a canonical isomorphism

$$(2.2) \quad D_0^{(r)}(B/A, M) \xrightarrow{\cong} \Omega_{B/A}^r \otimes_B M.$$

Let A be a k -algebra as above and let $\{B_n\}_{n \geq 0}$ be an inverse system of A -algebras. We denote this inverse system by B_\bullet . A B_\bullet -module is an inverse system $\{M^n\}_{n \geq 0}$ of A -modules such that for each $n \geq 0$, M^n is in fact a B_n -module and the map $M^n \xrightarrow{f^n} M^{n-1}$ is B_n -linear such that these maps are compatible with maps in the inverse system B_\bullet . A typical example in which we shall be mostly interested in is when I is an ideal of A and $M^n = B_n = A/I^{n+1}$.

Proposition 2.3. *Let B_\bullet be an inverse system of A -algebras. Let M_\bullet^\bullet and N_\bullet^\bullet be the flat simplicial B_\bullet -modules such that for each $q \geq 0$ and for each $n_0 \geq 0$, the map $H_q(M_\bullet^n) \rightarrow H_q(M_\bullet^{n_0})$ is zero for all $n \gg n_0$. Then for each $q \geq 0$ and for each $n_0 \geq 0$, the map $H_q(M_\bullet^n \otimes_{B_n} N_\bullet^n) \rightarrow H_q(M_\bullet^{n_0} \otimes_{B_{n_0}} N_\bullet^{n_0})$ is zero for all $n \gg n_0$.*

Proof. For any A algebra B , let $C : \text{SimpMod}(B) \rightarrow \text{Ch}_{\geq 0}(B)$ be the Dold-Kan functor from the category of simplicial B -modules to the category of chain complexes of B -modules which are bounded below at zero. This functor takes a simplicial module M_\bullet to itself and the differential at each level is the alternating sum of the face maps at that level. Then the Eilenberg-Zilber theorem (cf. [20], 1.6.12) implies that there is a natural Alexander-Whitney map

$$C(M_\bullet \otimes_B N_\bullet) \rightarrow C(M_\bullet) \otimes_B C(N_\bullet)$$

which is a quasi-isomorphism. Here the term on the right is the tensor product in the category of chain complexes, which is given as the total complex of the double complex $(M_\bullet, N_\bullet)_{i,j} = M_i \otimes_B N_j$. Hence it suffices to prove the lemma for the tensor product of chain complexes.

Since the double complex $\{M_i^n \otimes_{B_n} N_j^n\}_{i,j \geq 0}$ lies only in the first quadrant, there is a convergent spectral sequence ([28], 5.6.1)

$$\begin{aligned} {}^n E_{p,q}^2 &= H_p(\cdots \rightarrow H_q(M_\bullet^n \otimes_{B_n} N_\bullet^n) \rightarrow \cdots \rightarrow H_q(M_\bullet^n \otimes_{B_n} N_\bullet^1) \rightarrow H_q(M_\bullet^n \otimes_{B_n} N_\bullet^0)) \\ &\Rightarrow H_{p+q}(M_\bullet^n \otimes_{B_n} N_\bullet^n). \end{aligned}$$

This spectral sequence is compatible with the maps in the inverse systems $\{B_n\}$, $\{M^n\}$ and $\{N^n\}$ and we get an inverse system of spectral sequences

$$\begin{array}{ccc} {}^n E_{p,q}^2 & \Rightarrow & H_{p+q}(M_*^n \otimes_{B_n} N_*^n) \\ \downarrow & & \downarrow \\ {}^{n-1} E_{p,q}^2 & \Rightarrow & H_{p+q}(M_*^{n-1} \otimes_{B_{n-1}} N_*^{n-1}). \end{array}$$

Now as N_j^n is a flat B_n -module for each n and j , we see that ${}^n E_{p,q}^2$ is same as $H_p(H_q(M_*^n \otimes_{B_n} N_*^n))$. In particular, we see that for each $p, q \geq 0$ and for each $n_0 \geq 0$, the natural map ${}^n E_{p,q}^2 \rightarrow {}^{n_0} E_{p,q}^2$ is zero for all $n \gg n_0$ and hence the map ${}^n E_{p,q}^i \rightarrow {}^{n_0} E_{p,q}^i$ is zero for all $n \gg n_0$ and for all $i \geq 2$. From this we conclude that for a fixed $q \geq 0$, there is a filtration

$$0 = F_{-1}^n \subset F_0^n \subset \cdots \subset F_{q-1}^n \subset F_q^n (H_q(M_*^n \otimes_{B_n} N_*^n)) = H_q(M_*^n \otimes_{B_n} N_*^n)$$

and a map of filtered B_n -modules $H_q(M_*^n \otimes_{B_n} N_*^n) \rightarrow H_q(M_*^{n-1} \otimes_{B_{n-1}} N_*^{n-1})$ such that for each $j, n_0 \geq 0$, the map $F_j^n / F_{j-1}^n = {}^n E_{j,q-j}^\infty \rightarrow {}^{n_0} E_{j,q-j}^\infty = F_j^{n_0} / F_{j-1}^{n_0}$ is zero for all $n \gg n_0$.

Now we show by induction that for any $0 \leq j \leq q$ and any $n_0 \geq 0$, the map $F_j^n \rightarrow F_j^{n_0}$ is zero for all $n \gg n_0$. This will finish the proof of the proposition. The following trick (which we call the *doubling trick*) to do this will be used repeatedly in this paper. We fix j with $0 \leq j \leq q$ and by induction, assume that there exist $n_1 \gg n_0$ and $n_2 \gg n_1$ such that in the commutative diagram

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_{j-1}^n & \longrightarrow & F_j^n & \longrightarrow & {}^n E_{j,q-j}^\infty \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{j-1}^{n_1} & \longrightarrow & F_j^{n_1} & \longrightarrow & {}^{n_1} E_{j,q-j}^\infty \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{j-1}^{n_0} & \longrightarrow & F_j^{n_0} & \longrightarrow & {}^{n_0} E_{j,q-j}^\infty \longrightarrow 0, \end{array}$$

the bottom left and the bottom right vertical arrows are zero for all $n \geq n_1$, and the top left and the top right vertical arrows are zero for all $n \geq n_2$. A diagram chase shows that the composite middle vertical arrow is zero for all $n \geq n_2$. \square

Corollary 2.4. *Let B_\bullet be an inverse system of A -algebras and let M_\bullet^\bullet be a flat simplicial B_\bullet -module. Assume that for each $q \geq 0$ and for each $n_0 \geq 0$, the map $H_q(M_*^n) \rightarrow H_q(M_*^{n_0})$ is zero for all $n \gg n_0$. Then for each $r \geq 1$ and $q, n_0 \geq 0$, the map $H_q(\wedge_{B_n}^r M_*^n) \rightarrow H_q(\wedge_{B_{n_0}}^r M_*^{n_0})$ is zero for all $n \gg n_0$.*

Proof. By Lemma 2.2, it suffices to prove the corollary when the exterior powers are replaced by the corresponding tensor powers, when it follows directly from Proposition 2.3 and induction on r . \square

The following result was proved by Quillen ([23], Theorem 6.15) for $r = 1$.

Corollary 2.5. *Let A be an essentially of finite type k -algebra. Let I be an ideal of A and put $B_n = A/I^{n+1}$ for $n \geq 0$. Then for each $r \geq 1$ and $q, n_0 \geq 0$, the natural map $D_q^{(r)}(B_n/A) \rightarrow D_q^{(r)}(B_{n_0}/A)$ is zero for all $n \gg n_0$.*

Proof. We see from 2.1 that $\mathbb{L}_{B_n/A}$ is a free simplicial B_n -module. Since A is noetherian, we can apply [23] (Theorem 6.15) to conclude that for each $q, n_0 \geq 0$, there exists an N such that the natural map $D_q(B_{nn_0}/A, B_{n_0}) \rightarrow D_q(B_{n_0}/A, B_{n_0}) = D_q(B_{n_0}/A)$ is zero for all $n \geq N$. Since the map $D_q(B_{nn_0}/A) \rightarrow D_q(B_{n_0}/A)$ is the composite of the map

$$D_q(B_{nn_0}/A) = D_q(B_{nn_0}/A, B_{nn_0}) \rightarrow D_q(B_{nn_0}/A, B_{n_0}) \rightarrow D_q(B_{n_0}/A, B_{n_0}),$$

we see that the map $D_q(B_{nn_0}/A) \rightarrow D_q(B_{n_0}/A)$ is zero for all $n \geq N$. This in turn implies that the natural map $D_q(B_n/A) \rightarrow D_q(B_{n_0}/A)$ is zero for all $n \gg n_0$ (in fact for all $n \geq Nn_0$). Now we apply Corollary 2.4 to the inverse system $B_\bullet = \{B_n\}$ and the free simplicial module $M_\bullet^* = \{\mathbb{L}_{B_n/A}\}$ to conclude the proof of the corollary. \square

3. ARTIN-REES THEOREM FOR ANDRÉ-QUILLEN HOMOLOGY

Let k be a field and let A be a k -algebra which is essentially of finite type over k . Let I be an ideal of A and put $B_n = A/I^{n+1}$ for $n \geq 0$. Then $B_\bullet = \{B_n\}$ is an inverse system of finite A -algebras. Moreover, $\left\{ \frac{D_q^{(r)}(B_n/k)}{D_q^{(r)}(A/k)} \right\}_{n \geq 0}$ is a B_\bullet -module.

Similarly, $\left\{ \text{Ker} \left(D_q^{(r)}(A/k, B_n) \rightarrow D_q^{(r)}(B_n/k) \right) \right\}_{n \geq 0}$ is also a B_\bullet -module. Our aim in this section is to prove an Artin-Rees type theorem for these two modules. We begin with the following elementary result.

Lemma 3.1. *Let A be any k -algebra and let*

$$0 \rightarrow M'_* \rightarrow M_* \rightarrow M''_* \rightarrow 0$$

be a short exact sequence of free simplicial A -modules. Then there exists a convergent spectral sequence

$$E_{p,q}^1 = H_{q-p} \left(\wedge^p M'_* \otimes_A \wedge^{r-p} M''_* \right) \Rightarrow H_{q-p} \left(\wedge^r M_* \right).$$

This spectral sequence is natural for morphisms of k -algebras and morphisms of short exact sequences of free simplicial modules.

Proof. Exactness of simplicial modules means that it is exact at each level and the exactness is compatible with the face and the degeneracy maps. For each $i \geq 0$, we can define a decreasing finite filtration on $\wedge^r M_i$ by defining $F^j \wedge^r M_i$ to be the A -submodule generated by the forms of the type

$$\{a_1 \wedge \cdots \wedge a_r | a_{i_1}, \dots, a_{i_j} \in M'_i \text{ for some } 1 \leq i_1 \leq \cdots \leq i_j \leq r\}.$$

Then we have

$$\wedge^r M_i = F^0 \wedge^r M_i \supseteq \cdots \supseteq F^r \wedge^r M_i \supseteq F^{r+1} \wedge^r M_i = 0$$

and it is easy to check that for $0 \leq j \leq r$, the map

$$\beta_i^j : \wedge^j M'_i \otimes_A \wedge^{r-j} M_i \rightarrow F^j \wedge^r M_i,$$

$\beta_i^j((a_1 \wedge \cdots \wedge a_j) \otimes (b_1 \wedge \cdots \wedge b_{r-j})) = a_1 \wedge \cdots \wedge a_r \wedge b_1 \wedge \cdots \wedge b_{r-j}$
 descends to an isomorphism of quotients

$$(3.1) \quad \beta_i^j : \wedge^j M'_i \otimes_A \wedge^{r-j} M''_i \xrightarrow{\cong} \frac{F^j \wedge^r M_i}{F^{j+1} \wedge^r M_i}.$$

We also see from the above definition of the filtration and the maps β_i^j that this filtration and the isomorphisms in 3.1 are compatible with the morphisms of short exact sequences. In particular, they are compatible with the face and the degeneracy maps. Thus we get a decreasing filtration $\{F^j \wedge^r M_*\}_{0 \leq j \leq r}$ of the simplicial module $\wedge^r M_*$ such that for each $0 \leq j \leq r$, there is a natural isomorphism

$$(3.2) \quad \wedge^j M'_* \otimes_A \wedge^{r-j} M''_* \xrightarrow{\cong} \frac{F^j \wedge^r M_*}{F^{j+1} \wedge^r M_*}.$$

This filtration on the simplicial module $\wedge^r M_*$ gives ([28], 5.5) a convergent spectral sequence

$$E_{p,q}^1 = H_{q-p} \left(\frac{F^p \wedge^r M_*}{F^{p+1} \wedge^r M_*} \right) \Rightarrow H_{q-p}(\wedge^r M_*)$$

with differential $E_{p,q}^1 \rightarrow E_{p+1,q}^1$. The isomorphism of 3.2 now completes the proof of the existence of the spectral sequence. The functoriality with the morphisms of k -algebras and morphisms of exact sequences of simplicial modules is clear from the definition of the filtration above, which is preserved under a morphism of exact sequences. \square

Corollary 3.2. *Let A be an essentially of finite type algebra over a field k and let $l \subset k$ be any subfield. Then there is a convergent spectral sequence*

$$E_{p,q}^1 = \Omega_{k/l}^p \otimes_k D_{q-p}^{(r-p)}(A/k) \Rightarrow D_{q-p}^{(r)}(A/l).$$

Proof. We have ([23], proof of Theorem 5.1) a short exact sequence of free simplicial A -modules

$$0 \rightarrow \mathbb{L}_{k/l} \otimes_k A \rightarrow \mathbb{L}_{A/l} \rightarrow \mathbb{L}_{A/k} \rightarrow 0.$$

Put $\mathbb{K}_{A/l} = \mathbb{L}_{k/l} \otimes_k A$. Then Lemma 3.1 gives us a convergent spectral sequence

$$(3.3) \quad E_{p,q}^1 = H_{q-p}(\wedge^p \mathbb{K}_{A/l} \otimes_A \wedge^{r-p} \mathbb{L}_{A/k}) \Rightarrow H_{q-p}(\wedge^r \mathbb{L}_{A/l}).$$

To identify the E^1 -terms, we see from the proof of Proposition 2.3 that for each $p, q \geq 0$, there is a convergent spectral sequence

$$'E_{i,j}^2 = H_i(H_j(\wedge^p \mathbb{K}_{A/l}) \otimes_A \wedge^{r-p} \mathbb{L}_{A/k}) \Rightarrow H_{i+j}(\wedge^p \mathbb{K}_{A/l} \otimes_A \wedge^{r-p} \mathbb{L}_{A/k}).$$

Since A is k -flat, we have

$$H_j(\wedge^p \mathbb{K}_{A/l}) = H_j(\wedge^p \mathbb{L}_{k/l} \otimes_k A) = H_j(\wedge^p \mathbb{L}_{k/l}) \otimes_k A = D_j^{(p)}(k/l) \otimes_k A.$$

Since this last group is a free A -module, we obtain

$$'E_{i,j}^2 = D_j^{(p)}(k/l) \otimes_k D_i^{(r-p)}(A/k).$$

Now as k is a direct limit of its subfields which are finitely generated over l , and since the André-Quillen homology commutes with direct limits ([23], 4.11), we see

that $D_j^{(p)}(k/l)$ is a direct limit of the André-Quillen homology of the subfields of k which are finitely generated over l . In particular, we have ([20], Theorem 3.5.6)

$$D_j^{(p)}(k/l) = \begin{cases} \Omega_{k/l}^p & \text{if } p \geq 0, j = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus we get

$$'E_{i,j}^2 = \begin{cases} \Omega_{k/l}^p \otimes_k D_i^{(r-p)}(A/k) & \text{if } i \geq 0, j = 0 \\ 0 & \text{if } j > 0 \end{cases}.$$

In particular, this spectral sequence degenerates at $'E^2$ and we get for $p, i \geq 0$,

$$H_i(\wedge^p \mathbb{K}_{A/l} \otimes_A \wedge^{r-p} \mathbb{L}_{A/k}) = \Omega_{k/l}^p \otimes_k D_i^{(r-p)}(A/k).$$

Putting this in our spectral sequence of 3.3, we get the proof of the corollary. \square

Let A be an essentially of finite type algebra over a field k and let $B_\bullet = \{B_n = A/I^{n+1}\}$ be the inverse system of finite A -algebras as defined in the beginning of this section.

Lemma 3.3. *For any given $r, q, n_0 \geq 0$, the natural map*

$$\frac{D_q^{(r)}(A/k, B_n)}{D_q^{(r)}(A/k)} \rightarrow \frac{D_q^{(r)}(A/k, B_{n_0})}{D_q^{(r)}(A/k)}$$

is zero for all $n \gg n_0$.

Proof. For $r = 0$, both sides are zero, so we can assume $r \geq 1$. We first observe that for any $n \geq 0$, one has

$$\frac{D_q^{(r)}(A/k, B_n)}{D_q^{(r)}(A/k)} \xrightarrow{\cong} \frac{D_q^{(r)}(A/k, B_n)}{D_q^{(r)}(A/k) \otimes_A B_n}.$$

By [23] (4.7), there is a convergent spectral sequence

$${}^n E_{p,q}^2 = \text{Tor}_p^A(D_q^{(r)}(A/k), B_n) \Rightarrow D_{p+q}^{(r)}(A/k, B_n).$$

This spectral sequence is compatible with the maps $B_n \twoheadrightarrow B_{n-1}$ and gives a finite filtration of $D_q^{(r)}(A/k, B_n)$

$$0 = F_{-1}^n \subseteq F_0^n \subseteq \cdots \subseteq F_{q-1}^n \subseteq F_q^n = D_q^{(r)}(A/k, B_n)$$

such that ${}^n E_{j,q-j}^\infty = F_j^n / F_{j-1}^n$ for $0 \leq j \leq q$ and the edge map gives $D_q^{(r)}(A/k) \otimes_A B_n \twoheadrightarrow F_0^n$. Hence it suffices to show that the natural map

$$(3.4) \quad \frac{D_q^{(r)}(A/k, B_n)}{F_0^n D_q^{(r)}(A/k, B_n)} \rightarrow \frac{D_q^{(r)}(A/k, B_{n_0})}{F_0^{n_0} D_q^{(r)}(A/k, B_{n_0})}$$

is zero for all $n \gg n_0$. Using the above filtration, an induction on j and the doubling trick of 2.3, this is reduced to showing that for $1 \leq j \leq q$, the map ${}^n E_{j,q-j}^2 \rightarrow {}^{n_0} E_{j,q-j}^2$ is zero for all $n \gg n_0$. But for $j \geq 1$, we have ${}^n E_{j,q-j}^2 =$

$\mathrm{Tor}_j^A \left(D_{q-j}^{(r)}(A/k), B_n \right)$. Furthermore, A is a localization of a finite type k -algebra and so by [23] (Proposition 4.12, Theorem 5.4(i)), $D_q^{(r)}(A/k)$ is a finite A -module for all $r, q \geq 0$. Hence by [1] (Proposition 10, Lemma 11), the map

$$\mathrm{Tor}_j^A \left(D_{q-j}^{(r)}(A/k), B_n \right) \rightarrow \mathrm{Tor}_j^A \left(D_{q-j}^{(r)}(A/k), B_{n_0} \right)$$

is zero for all $n \gg n_0$. \square

Proposition 3.4. *Let A and B_\bullet be as above. Then for any given $r, q, n_0 \geq 0$, the natural map*

$$\frac{D_q^{(r)}(B_n/k)}{D_q^{(r)}(A/k)} \rightarrow \frac{D_q^{(r)}(B_{n_0}/k)}{D_q^{(r)}(A/k)}$$

is zero for all $n \gg n_0$.

Proof. For $r = 0$, both sides are zero, so we assume $r \geq 1$. Since the map $D_q^{(r)}(A/k) \rightarrow D_q^{(r)}(B_n/k)$ factors through the map $D_q^{(r)}(A/k) \rightarrow D_q^{(r)}(A/k, B_n)$, one has for all n , the natural exact sequence

$$(3.5) \quad \frac{D_q^{(r)}(A/k, B_n)}{D_q^{(r)}(A/k)} \rightarrow \frac{D_q^{(r)}(B_n/k)}{D_q^{(r)}(A/k)} \rightarrow \frac{D_q^{(r)}(B_n/k)}{D_q^{(r)}(A/k, B_n)} \rightarrow 0.$$

Using the doubling trick of 2.3 and Lemma 3.3, we only need to show that for any given $r \geq 1$ and $q, n_0 \geq 0$, the natural map

$$(3.6) \quad \frac{D_q^{(r)}(B_n/k)}{D_q^{(r)}(A/k, B_n)} \rightarrow \frac{D_q^{(r)}(B_{n_0}/k)}{D_q^{(r)}(A/k, B_{n_0})}$$

is zero for all $n \gg n_0$.

For $n \geq 0$, we put $\mathbb{K}_{B_n/k} = \mathbb{L}_{A/k} \otimes_A B_n$. Then we observe that for $r, q \geq 0$, $D_q^{(r)}(A/k, B_n)$ is same as $H_q \left((\wedge_A^r \mathbb{L}_{A/k}) \otimes_A B_n \right) = H_q \left(\wedge_{B_n}^r \mathbb{K}_{B_n/k} \right)$. For all $n \geq 0$, we have an exact sequence ([23], Theorem 5.1) of free simplicial B_n -modules

$$0 \rightarrow \mathbb{L}_{A/k} \otimes_A B_n \rightarrow \mathbb{L}_{B_n/k} \rightarrow \mathbb{L}_{B_n/A} \rightarrow 0.$$

Hence by Lemma 3.1, there is a convergent spectral sequence

$${}^n E_{p,q}^1 = H_{q-p} \left(\wedge^p \mathbb{K}_{B_n/k} \otimes_{B_n} \wedge^{r-p} \mathbb{L}_{B_n/A} \right) \Rightarrow H_{q-p} \left(\wedge^r \mathbb{L}_{B_n/k} \right).$$

This spectral sequence is compatible with the maps $B_n \twoheadrightarrow B_{n-1}$ and gives a finite filtration of $H_q \left(\wedge^r \mathbb{L}_{B_n/k} \right)$

$$(3.7) \quad H_q \left(\wedge^r \mathbb{L}_{B_n/k} \right) = F_0^n \supseteq F_1^n \supseteq \cdots \supseteq F_r^n \supseteq F_{r+1}^n = 0$$

with $F_j^n / F_{j+1}^n \cong {}^n E_{j,q+j}^\infty$ for $0 \leq j \leq r$ and a morphism of filtered modules $H_q \left(\wedge^r \mathbb{L}_{B_n/k} \right) \rightarrow H_q \left(\wedge^r \mathbb{L}_{B_{n-1}/k} \right)$. Furthermore, the edge map gives a surjection $H_q \left(\wedge^r \mathbb{K}_{B_n/k} \right) \twoheadrightarrow F_r^n H_q \left(\wedge^r \mathbb{L}_{B_n/k} \right)$. In particular, we have $\frac{H_q \left(\wedge^r \mathbb{L}_{B_n/k} \right)}{H_q \left(\wedge^r \mathbb{K}_{B_n/k} \right)} \xrightarrow{\cong}$

$\frac{H_q(\wedge^r \mathbb{L}_{B_n/k})}{F_r^n H_q(\wedge^r \mathbb{L}_{B_n/k})}$. Hence using the filtration in 3.7, an induction on j and the doubling trick of 2.3, it suffices to show that for $0 \leq j \leq r-1$, the natural map ${}^n E_{j,q+j}^1 \rightarrow {}^{n_0} E_{j,q+j}^1$ is zero for all $n \gg n_0$. But this follows from Proposition 2.3 and Corollary 2.5. This proves 3.6 and hence the proposition. \square

Lemma 3.5. *Let the k -algebras A and B_\bullet be as above. Then for any given $r, q, n_0 \geq 0$, the natural map*

$$\text{Ker} \left(D_q^{(r)}(A/k, B_n) \rightarrow D_q^{(r)}(B_n/k) \right) \rightarrow \text{Ker} \left(D_q^{(r)}(A/k, B_{n_0}) \rightarrow D_q^{(r)}(B_{n_0}/k) \right)$$

is zero for all $n \gg n_0$.

Proof. For $r = 0$, both sides are zero, so we assume $r \geq 1$. We have seen in the proof of Proposition 3.4 that for all $n \geq 0$, $D_q^{(r)}(B_n/k) = H_q(\wedge^r \mathbb{L}_{B_n/k})$ has a finite filtration $\{F_j^n\}_{0 \leq j \leq r}$ such that $H_q(\wedge^r \mathbb{L}_{B_n/k}) \twoheadrightarrow F_r^n H_q(\wedge^r \mathbb{L}_{B_n/k})$. Thus we can replace $D_q^{(r)}(B_n/k)$ by $F_r^n H_q(\wedge^r \mathbb{L}_{B_n/k}) = {}^n E_{r,q+r}^\infty = {}^n E_{r,q+r}^{r+1}$ in the statement of the lemma.

Now for $0 \leq j \leq r$, there is an exact sequence

$${}^n E_{r-j-1,q+r+j}^{j+1} \rightarrow {}^n E_{r,q+r}^{j+1} \rightarrow {}^n E_{r,q+r}^{j+2} \rightarrow 0,$$

which gives a finite increasing filtration of $\text{Ker} \left(D_q^{(r)}(A/k, B_n) = {}^n E_{r,q+r}^1 \rightarrow {}^n E_{r,q+r}^{r+1} \right)$

$$0 = \overline{F}_{-1}^n \subseteq \overline{F}_0^n \subseteq \cdots \subseteq \overline{F}_{r-1}^n \subseteq \overline{F}_r^n = \text{Ker} \left({}^n E_{r,q+r}^1 \rightarrow {}^n E_{r,q+r}^{r+1} \right)$$

such that for all $0 \leq j \leq r$,

$${}^n E_{r-j-1,q+r+j}^{j+1} \twoheadrightarrow \overline{F}_j^n / \overline{F}_{j-1}^n.$$

Thus by using an induction on j and the doubling trick as before, it suffices to show that for $0 \leq j \leq r$ and for given $n_0 \geq 0$, the natural map ${}^n E_{r-j-1,q+r+j}^1 \rightarrow {}^{n_0} E_{r-j-1,q+r+j}^1$ is zero for all $n \gg n_0$. But this again follows from Proposition 2.3 and Corollary 2.5. \square

4. ANDRÉ-QUILLEN HOMOLOGY AND NORMALIZATION I

Let A be an integral domain which is essentially of finite type algebra over a field k . Assume that A is singular, and let $f : A \rightarrow B$ be the normalization morphism of A . We assume that B is smooth over k . Our aim in this section is to estimate the André-Quillen homology of the conducting ideals for this normalization. We begin with estimating the kernels of the maps between differential forms. For any conducting ideal $I \subset A$ for the normalization and for $r \geq 0$, let $\Omega_{(A,I)/l}^r$ (resp $\Omega_{(B,I)/l}^r$) denote the kernel of the map $\Omega_{A/l}^r \twoheadrightarrow \Omega_{(A/I)/l}^r$ (resp $\Omega_{B/l}^r \twoheadrightarrow \Omega_{(B/I)/l}^r$) for any subfield $l \subset k$.

Lemma 4.1. *Let A and B be as above. Then for any given conducting ideal $I \subset A$ for the normalization and for any $r \geq 0$, the natural map*

$$\Omega_{(A,I^n)/k}^r \rightarrow \Omega_{(B,I^n)/k}^r$$

is injective for all sufficiently large n .

Proof. We see from 2.2 and Lemma 3.5 that for any given $n_0 \geq 0$ the natural map

$$(4.1) \quad \text{Ker} \left(\Omega_{A/k}^r \otimes_A A/I^n \xrightarrow{u_n^A} \Omega_{(A/I^n)/k}^r \right) \rightarrow \text{Ker} \left(\Omega_{A/k}^r \otimes_A A/I^{n_0} \xrightarrow{u_{n_0}^A} \Omega_{(A/I^{n_0})/k}^r \right)$$

is zero for all $n \gg n_0$. In the same way, the map $\text{Ker}(u_n^B) \rightarrow \text{Ker}(u_{n_0}^B)$ is zero for all $n \gg n_0$. On the other hand, we have a commutative diagram of exact sequences for any $n \geq 0$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^n \Omega_{A/k}^r & \longrightarrow & \Omega_{(A,I^n)/k}^r & \longrightarrow & \text{Ker}(u_n^A) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^n \Omega_{B/k}^r & \longrightarrow & \Omega_{(B,I^n)/k}^r & \longrightarrow & \text{Ker}(u_n^B) \longrightarrow 0 \end{array}$$

We claim that the map $I^n \Omega_{A/k}^r \rightarrow I^n \Omega_{B/k}^r$ is injective for all $n \gg 0$. To see this, note that $\Omega_{A/k}^r$ is a finite A -module and we can apply the Artin-Rees theorem to find a $c > 0$ such that all $n > c$, one has

$$(I^n \Omega_{A/k}^r \cap \Omega_{(A,B)/k}^r) \subseteq I^{n-c} (I^c \Omega_{A/k}^r \cap \Omega_{(A,B)/k}^r) \subseteq I^{n-c} \Omega_{(A,B)/k}^r,$$

where $\Omega_{(A,B)/k}^r = \text{Ker}(\Omega_{A/k}^r \rightarrow \Omega_{B/k}^r)$. On the other hand, the finite A -module $\Omega_{(A,B)/k}^r$ is supported on the support of I and hence $I^{n-c} \Omega_{(A,B)/k}^r = 0$ for all $n \gg 0$. This proves the claim. Using the claim in the above diagram, we see that there exists an n_0 such that $\text{Ker}(\Omega_{(A,I^n)/k}^r \rightarrow \Omega_{(B,I^n)/k}^r) \hookrightarrow \text{Ker}(u_n^A)$ for all $n \geq n_0$. Now we use 4.1 to conclude that the map

$$\text{Ker}(\Omega_{(A,I^n)/k}^r \rightarrow \Omega_{(B,I^n)/k}^r) \rightarrow \text{Ker}(\Omega_{(A,I^{n_0})/k}^r \rightarrow \Omega_{(B,I^{n_0})/k}^r)$$

is zero for all $n \gg n_0$. However, this map is clearly injective. Hence we must have $\text{Ker}(\Omega_{(A,I^n)/k}^r \rightarrow \Omega_{(B,I^n)/k}^r) = 0$ for all $n \gg 0$. \square

Lemma 4.2. *Let $f : A \rightarrow B$ be as above. Then for any conducting ideal I for the normalization and for any $r, q \geq 1$, the natural map*

$$D_q^{(r)}(A/k) \xrightarrow{u_n^A} D_q^{(r)}(A/I^n/k)$$

is injective for all $n \gg 0$.

Proof. We first observe that $D_q^{(r)}(B/k) = 0$ for $q \geq 1$ as B is smooth ([20], Theorem 3.5.6). Since the homology groups $D_q^{(r)}(A/k)$ are finite A -modules and since they commute with the localization ([23], Theorem 5.4 (i)), we see as before that $I^n D_q^{(r)}(A/k) = 0$ for all $n \gg 0$ whenever $q \geq 1$. In particular, for any $r, q \geq 1$ there exists $N \gg 0$ such that for all $n \geq N$,

$$(4.2) \quad D_q^{(r)}(A/k) \xrightarrow{\cong} D_q^{(r)}(A/k) \otimes_A A/I^n.$$

For $r, q \geq 1$ and $n \geq 0$, we have a convergent spectral sequence

$${}^n E_{p,q}^2 = \text{Tor}_p^A(D_q^{(r)}(A/k), A/I^n) \Rightarrow D_{p+q}^{(r)}(A/k, A/I^n)$$

with differential ${}^n E_{p,q}^2 \rightarrow {}^n E_{p-2,q+1}^2$. This gives a filtration

$$0 = F_{-1}^n \subseteq F_0^n \subseteq \cdots \subseteq F_{q-1}^n \subseteq F_q^n = D_q^{(r)}(A/k, A/I^n)$$

and map of filtered modules $D_q^{(r)}(A/k, A/I^n) \rightarrow D_q^{(r)}(A/k, A/I^{n-1})$. The edge map further gives a surjection $D_q^{(r)}(A/k) \otimes_A A/I^n = {}^n E_{0,q}^2 \twoheadrightarrow F_0^n = {}^n E_{0,q}^\infty = {}^n E_{0,q}^{q+2}$.

We now show that for $r, q \geq 1$ and $n_0 \geq 0$, the natural map

$$(4.3) \quad \begin{aligned} &\text{Ker}(D_q^{(r)}(A/k) \otimes_A A/I^n \rightarrow F_0^n D_q^{(r)}(A/k, A/I^n)) \\ &\rightarrow \text{Ker}(D_q^{(r)}(A/k) \otimes_A A/I^{n_0} \rightarrow F_0^{n_0} D_q^{(r)}(A/k, A/I^{n_0})) \end{aligned}$$

is zero for all $n \gg n_0$.

For $0 \leq j \leq q$, there is an exact sequence

$${}^n E_{j+2,q-j-1}^{j+2} \rightarrow {}^n E_{0,q}^{j+2} \rightarrow {}^n E_{0,q}^{j+3} \rightarrow 0.$$

Thus by letting $\Gamma_j^n = \text{Ker}({}^n E_{0,q}^{j+2} \rightarrow {}^n E_{0,q}^{j+3})$ for $0 \leq j \leq q$, we get a filtration $0 = \Gamma_0^n \subseteq \Gamma_1^n \subseteq \cdots \subseteq \Gamma_q^n$ of Γ_q^n such that

$${}^n E_{j+2,q-j-1}^{j+2} \twoheadrightarrow \Gamma_{j+1}^n / \Gamma_j^n.$$

Thus to prove 4.3, it suffices to show by an induction on j and the doubling trick that for given $r, q, n_0 \geq 1$ and for $0 \leq j \leq q$, the natural map ${}^n E_{j+2,q-j-1}^{j+2} \rightarrow {}^{n_0} E_{j+2,q-j-1}^{j+2}$ is zero for $n \gg n_0$. But for $n, j \geq 0$ we have ${}^n E_{j+2,q-j-1}^{j+2} = \text{Tor}_{j+2}^A(D_{q-j-1}^{(r)}(A/k), A/I^n)$ and the map

$$\text{Tor}_{j+2}^A(D_{q-j-1}^{(r)}(A/k), A/I^n) \rightarrow \text{Tor}_{j+2}^A(D_{q-j-1}^{(r)}(A/k), A/I^{n_0})$$

is zero by for $n \gg n_0$ by [1](Proposition 10, Lemma 11) as $D_q^{(r)}(A/k)$ are all finite A -modules. This proves 4.3.

If $D_q^{(r)}(A/k) \xrightarrow{\theta_n^A} D_q^{(r)}(A/k, A/I^n)$ and $D_q^{(r)}(A/k, A/I^n) \xrightarrow{v_n^A} D_q^{(r)}(A/I^n/k)$ denote the natural maps, then for any $r, q, n \geq 1$, we get a natural exact sequence

$$0 \rightarrow \text{Ker}(\theta_n^A) \rightarrow \text{Ker}(u_n^A) \rightarrow \text{Ker}(v_n^A).$$

Now for any $n_0 \geq N$, we can use 4.2 and 4.3 to conclude that the map $\text{Ker}(\theta_n^A) \rightarrow \text{Ker}(\theta_{n_0}^A)$ is zero for all $n \gg n_0$. The map $\text{Ker}(v_n^A) \rightarrow \text{Ker}(v_{n_0}^A)$ is zero for all $n \gg n_0$ by Lemma 3.5. The doubling trick again shows that for any $r, q \geq 1$ and $n_0 \geq N$, the natural map $\text{Ker}(u_n^A) \rightarrow \text{Ker}(u_{n_0}^A)$ is zero for all $n \gg n_0$. But this last map is clearly injective. Hence we must have $\text{Ker}(u_n^A) = 0$ for all $n \gg 0$. \square

5. ANDRÉ-QUILLEN HOMOLOGY AND NORMALIZATION II

Most of the proofs in the previous section relied on the fact that the algebra A is essentially of *finite type* over the field k . Our aim in this section is to generalize the results of the previous section to the case when the base ring for the André-Quillen homology of k -algebras is any subfield of k . Our eventual application will need these results when the base ring is the field of rational numbers. So let A be an integral domain which is essentially of finite type algebra over a field k . Let $f : A \rightarrow B$ be the normalization of A such that B is smooth. Let $l \subset k$ be any subfield.

The basic extra ingredient to deal with the general case will be our spectral sequence of Corollary 3.2:

$$(5.1) \quad E_{p,q}^1(A) = \Omega_{k/l}^p \otimes_k D_{q-p}^{(r-p)}(A/k) \Rightarrow D_{q-p}^{(r)}(A/l).$$

As shown in Lemma 3.1, this spectral sequence is clearly compatible with the morphisms of k -algebras. We denote the corresponding spectral sequence for A/I^n by ${}^n E_{p,q}^i$ as before and that for B by $E_{p,q}^i(B)$. Put $E_{p,q}^i(A, B) = \text{Ker}(E_{p,q}^i(A) \rightarrow E_{p,q}^i(B))$.

Lemma 5.1. *For any $r, i \geq 1$ and $p, q, n_0 \geq 0$, the natural map*

$$(5.2) \quad \frac{{}^n E_{p,q}^i}{E_{p,q}^i(A)} \rightarrow \frac{{}^{n_0} E_{p,q}^i}{E_{p,q}^i(A)}$$

is zero for all $n \gg n_0$. Furthermore, the natural map

$$(5.3) \quad E_{p,q}^i(A, B) \rightarrow {}^n E_{p,q}^i$$

is injective for all $n \gg n_0$.

Proof. We prove both statements by induction on $i \geq 1$. For $i = 1$, 5.2 follows directly from Proposition 3.4 and 5.3 follows directly from Lemmas 4.1 and 4.2. So assume that 5.2 and 5.3 hold for all $1 \leq j \leq i$. We first show that 5.3 holds for $i + 1$. Consider the following commutative diagram of exact sequences.

$$\begin{array}{ccccccc} E_{p-i, q+i-1}^i(A) & \longrightarrow & \text{Ker}(\partial_{p,q}^i(A)) & \longrightarrow & E_{p,q}^{i+1}(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ E_{p-i, q+i-1}^i(B) & \longrightarrow & \text{Ker}(\partial_{p,q}^i(B)) & \longrightarrow & E_{p,q}^{i+1}(B) & \longrightarrow & 0, \end{array}$$

where $E_{p,q}^i \xrightarrow{\partial_{p,q}^i} E_{p+i, q-i+1}^i$ is the differential of the spectral sequence. If $p \neq q$, then $E_{p,q}^1(B) = \Omega_{k/l}^p \otimes_k D_{q-p}^{(r-p)}(B/k) = 0$ (as B is smooth over k) and so is $E_{p,q}^j(B)$ for $j \geq 1$. This gives exact sequence

$$(5.4) \quad E_{p-i, q+i-1}^i(A) \rightarrow \text{Ker}(\text{Ker}(\partial_{p,q}^i(A)) \rightarrow \text{Ker}(\partial_{p,q}^i(B))) \rightarrow E_{p,q}^{i+1}(A, B) \rightarrow 0.$$

If $p = q$, then $q + i - 1 - p + i = 2i - 1 \geq 1$ and we get $E_{p-i, q+i-1}^i(B) = 0$. This again gives the exact sequence as above. Thus 5.4 holds for all $p, q \geq 0$. Now we

consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
E_{p-i, q+i-1}^i(A) & \rightarrow & \text{Ker}(\text{Ker}(\partial_{p,q}^i(A)) \rightarrow \text{Ker}(\partial_{p,q}^i(B))) & \rightarrow & E_{p,q}^{i+1}(A, B) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
{}^n E_{p-i, q+i-1}^i & \longrightarrow & \text{Ker}({}^n \partial_{p,q}^i) & \longrightarrow & {}^n E_{p,q}^{i+1} & \longrightarrow & 0
\end{array}$$

The middle vertical arrow is injective for $n \gg n_0$ by induction. Let N be the smallest integer such that this arrow is injective for $n \geq N$. Then a diagram chase shows that $\text{Ker}(E_{p,q}^{i+1}(A, B) \rightarrow {}^n E_{p,q}^{i+1})$ is an A -submodule of a quotient of $\frac{{}^n E_{p-i, q+i-1}^i}{E_{p-i, q+i-1}^i(A)}$ for all $n \geq N$. Thus to prove 5.3 for $i+1$, it suffices to show that the natural map

$$\frac{{}^n E_{p-i, q+i-1}^i}{E_{p-i, q+i-1}^i(A)} \rightarrow \frac{{}^N E_{p-i, q+i-1}^i}{E_{p-i, q+i-1}^i(A)}$$

is zero for all $n \gg N$. But this is true as 5.2 holds for i by induction.

Now we show that 5.2 holds for $i+1$. We have a commutative diagram

$$\begin{array}{ccc}
E_{p,q}^i(A) & \xrightarrow{\partial_{p,q}^i(A)} & E_{p+i, q-i+1}^i(A) \\
\downarrow & & \downarrow \\
E_{p,q}^i(B) & \xrightarrow{\partial_{p,q}^i(B)} & E_{p+i, q-i+1}^i(B).
\end{array}$$

If $p \neq q$, then $E_{p,q}^1(B) = 0 = E_{p,q}^i(B)$ and if $p = q$, then $q-i+1-p-i = 1-2i < 0$ as $i \geq 1$ and hence $E_{p+i, q-i+1}^i(B) = 0$. Now the above diagram shows that for $p, q \geq 0$, one has a factorization

$$(5.5) \quad E_{p,q}^i(A) \xrightarrow{\partial_{p,q}^i(A)} E_{p+i, q-i+1}^i(A, B) \hookrightarrow E_{p+i, q-i+1}^i(A).$$

Next we consider another commutative diagram for $n \geq 0$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker}(\partial_{p,q}^i(A)) & \longrightarrow & E_{p,q}^i(A) & \longrightarrow & \text{Image}(\partial_{p,q}^i(A)) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}({}^n \partial_{p,q}^i) & \longrightarrow & {}^n E_{p,q}^i & \longrightarrow & \text{Image}({}^n \partial_{p,q}^i) \longrightarrow 0
\end{array}$$

Since 5.3 holds for i , we see from 5.5 that the right vertical arrow is injective for all $n \gg n_0$. In particular, we get an inclusion

$$\frac{\text{Ker}({}^n \partial_{p,q}^i)}{\text{Ker}(\partial_{p,q}^i(A))} \hookrightarrow \frac{{}^n E_{p,q}^i}{E_{p,q}^i(A)}$$

for all $n \geq N \gg n_0$. We now apply induction on i in 5.2 and this inclusion to conclude that the map

$$\frac{\text{Ker}({}^n\partial_{p,q}^i)}{\text{Ker}(\partial_{p,q}^i(A))} \rightarrow \frac{\text{Ker}({}^N\partial_{p,q}^i)}{\text{Ker}(\partial_{p,q}^i(A))}$$

is zero for all $n \gg N$. Since $\frac{{}^nE_{p,q}^{i+1}}{E_{p,q}^{i+1}(A)}$ is a quotient of the module on the left for $n \geq 0$, we get that the map

$$\frac{{}^nE_{p,q}^{i+1}}{E_{p,q}^{i+1}(A)} \rightarrow \frac{{}^NE_{p,q}^{i+1}}{E_{p,q}^{i+1}(A)}$$

is zero for all $n \gg N$. Since $N \geq n_0$, we see that 5.2 holds for $i \geq 1$. This proves the lemma. \square

The following is the generalization of Proposition 3.4 and Lemmas 4.1 and 4.2 to the case when the base ring of the André-Quillen homology is any subfield of the given field k .

Proposition 5.2. *Let $f : A \rightarrow B$ be the smooth normalization of an essentially of finite type k -algebra A . Let $l \subset k$ be a subfield. Let $I \subset A$ be any given conducting ideal for the normalization. Then*

(i) *For any $r, q, n_0 \geq 0$, the natural map*

$$\frac{D_q^{(r)}(A/I^n/l)}{D_q^{(r)}(A/l)} \rightarrow \frac{D_q^{(r)}(A/I^{n_0}/l)}{D_q^{(r)}(A/l)}$$

is zero for all $n \gg n_0$.

(ii) *For any $r, q, n_0 \geq 0$, the natural map*

$$\frac{D_q^{(r)}(B/I^n/l)}{D_q^{(r)}(B/l)} \rightarrow \frac{D_q^{(r)}(B/I^{n_0}/l)}{D_q^{(r)}(B/l)}$$

is zero for all $n \gg n_0$.

(iii) *For any $r, q \geq 1$, the natural maps*

$$D_q^{(r)}(A/l) \rightarrow D_q^{(r)}(A/I^n/l)$$

$$D_0^{(r)}((A, B)/l) \rightarrow D_q^{(r)}(A/I^n/l)$$

are injective for all $n \gg n_0$.

Proof. For $r = 0$, the part (i) is obvious as the groups on the both sides are zero. So we assume $r \geq 1$. The spectral sequence 5.1 (cf. Corollary 3.2) gives for any $r \geq 1$ and $q \geq 0$, a finite filtration of $D_q^{(r)}(A/l)$

$$D_q^{(r)}(A/l) = F^0(A) \supseteq F^1(A) \supseteq \cdots \supseteq F^r(A) \supseteq F^{r+1}(A) = 0$$

with $F^j(A)/F^{j+1}(A) \cong E_{j,q+j}^\infty(A) = E_{j,q+j}^{r+1}(A)$ for $0 \leq j \leq r$. One has similar filtrations for $D_q^{(r)}(B/l)$ and $D_q^{(r)}(A/I^n/l)$ together with morphisms of filtered A -modules. Thus for $0 \leq j \leq r$ and $n \geq 0$, we have the exact sequence

$$\frac{F^{j+1}(A/I^n)}{F^{j+1}(A)} \rightarrow \frac{F^j(A/I^n)}{F^j(A)} \rightarrow \frac{{}^n E_{j,q+j}^{r+1}}{E_{j,q+j}^{r+1}(A)} \rightarrow 0.$$

Now by comparing this exact sequence for n_0 and $n \geq n_0$, using the doubling trick in 2.3 as before and using the descending induction on j , we see that it suffices to show that for $r \geq 1$, $q, n_0 \geq 0$ and $0 \leq j \leq r$, the natural map

$$\frac{{}^n E_{j,q+j}^{r+1}}{E_{j,q+j}^{r+1}(A)} \rightarrow \frac{{}^{n_0} E_{j,q+j}^{r+1}}{E_{j,q+j}^{r+1}(A)}$$

is zero for all $n \gg n_0$ to prove part (i) of the proposition. But this follows from Lemma 5.1.

To prove (ii), we first observe from the smoothness of B and the above spectral sequence that

$$D_q^{(r)}(B/l) = \begin{cases} \Omega_{B/l}^r & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Furthermore, as $D_0^{(r)}(B/I^n/l) = \Omega_{(B/I^n)/l}^r$ (cf. 2.2), we immediately get (ii) for $q = 0$. We also get from this that for $q \geq 1$ and $n \geq 0$,

$$(5.6) \quad \frac{D_q^{(r)}(B/I^n/l)}{D_q^{(r)}(B/l)} \xrightarrow{\cong} D_q^{(r)}(B/I^n/l).$$

Thus we need to show that the map $D_q^{(r)}(B/I^n/l) \rightarrow D_q^{(r)}(B/I^{n_0}/l)$ is zero for all $n \gg n_0$. For $r, q \geq 1$, the natural map

$$(5.7) \quad D_q^{(r)}(B/I^n/k) \rightarrow D_q^{(r)}(B/I^{n_0}/k)$$

is zero for all $n \gg n_0$ by 5.6 and Proposition 3.4. Now the proof of (ii) follows by using 5.7 and the spectral sequence 5.1 and then by following exactly the same argument as in the proof of (i). Here the analogue of Lemma 5.1 follows immediately from 5.1 and 5.7.

For proving part (iii), we consider the following diagram of exact sequences for $0 \leq j \leq r$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^{j+1}(A, B) & \longrightarrow & F^j(A, B) & \longrightarrow & E_{j,q+j}^{r+1}(A, B) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}^n F^{j+1} & \longrightarrow & {}^n F^j & \longrightarrow & {}^n E_{j,q+j}^{r+1} \longrightarrow 0, \end{array}$$

where $F^j(A, B) = \text{Ker}(F^j(A) \rightarrow F^j(B))$ and ${}^n F^j$ is the filtration of $D_q^{(r)}(A/I^n/l)$. Again by descending induction on j , it suffices to show that for $r \geq 1$, $q, n_0 \geq 0$ and for $0 \leq j \leq r$, the map

$$E_{j,q+j}^{r+1}(A, B) \rightarrow {}^n E_{j,q+j}^{r+1}$$

is injective. But this follows again from Lemma 5.1. \square

6. ANDRÉ-QUILLEN TO HOCHSCHILD HOMOLOGY

In this section, we derive some consequences of our results of the previous section for the Hochschild and cyclic homology. Let k be a field and let A be an essentially of finite type k -algebra. We recall that for any A -algebra B , the Hochschild homology $HH_*^A(B)$ are the homology group of the pre-simplicial B -module $C_*^A(B)$ given by

$$C_n^A(B) = B \otimes_A \cdots \otimes_A B = B^{\otimes_{n+1}} \text{ and the face maps}$$

$d_i : C_n^A(B) \rightarrow C_{n-1}^A(B)$ for $0 \leq i \leq n$ being given by

$$d_i(a_0, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n) \text{ for } 0 \leq i \leq n-1 \text{ and}$$

$$d_n(a_0, \dots, a_n) = (a_n a_0, a_1, \dots, a_{n-1}).$$

The associated chain complex of B -modules is called the *Hochschild complex* of B over A .

To define the cyclic homology of B , one uses the natural action of finite cyclic groups on the Hochschild complex to construct the cyclic bicomplex $CC^A(B)$ and the cyclic homology $HC_*^A(B)$ of B over A are defined as the homology of the associated total complex. We refer the reader to [20] for the details about the definitions of Hochschild and cyclic homology and their properties which are relevant to us in this paper. For any map $B \rightarrow B'$ of A -algebras, the relative Hochschild homology $HH_*^A(B, B')$ is defined as the homology of the complex $\text{Cone}(HH_*^A(B) \rightarrow HH_*^A(B'))[-1]$. For an ideal $I \subset B$, the relative Hochschild homology $HH_*^A(B, I)$ is the relative homology of the map $B \rightarrow B/I$. For a map $B \rightarrow B'$ and an ideal $I \subset B$ such that $IB' = I$, the double relative Hochschild homology are defined as the homology of the complex $\text{Cone}(HH_*^A(B, I) \rightarrow HH_*^A(B', I))[-1]$. The relative and double relative cyclic homology are defined in the analogous way by taking the cones over the total cyclic complexes. If $CC^A(B)^2$ denotes the cyclic bicomplex of B consisting of only the first two columns of $CC^A(B)$, then there is a natural short exact sequence (*loc. cit.*, Theorem 2.2.1)

$$0 \rightarrow CC^A(B)^2 \rightarrow CC^A(B) \rightarrow CC^A(B)[2, 0] \rightarrow 0,$$

which gives the Connes' periodicity long exact sequence (also called SBI-sequence)

$$\cdots \rightarrow HH_n^A(B) \xrightarrow{I} HC_n^A(B) \xrightarrow{S} HC_{n-2}^A(B) \xrightarrow{B} HH_{n-1}^A(B) \xrightarrow{I} \cdots$$

Taking the cones over the appropriate short exact sequences as above, one gets similar SBI-sequence for the relative and double relative Hochschild and cyclic homology.

The most crucial fact which will be useful to us in this paper is the canonical decomposition of the Hochschild homology in terms of the André-Quillen homology in characteristic zero. We state it here for the sake of reader's convenience and refer to *loc. cit.* (Theorem 3.5.8) for the proof.

Theorem 6.1. *For any flat A -algebra B , there is a canonical decomposition*

$$HH_n^A(B) \cong \bigoplus_{r+q=n} D_q^{(r)}(B/A).$$

It is also known that this decomposition is compatible with the Hodge decomposition of the Hochschild homology.

An immediate consequence of this canonical decomposition is the following Hochschild homology analogue of Proposition 5.2. As in the previous section, let A be an integral domain which is an essentially of finite type algebra over a field k . Let $f : A \rightarrow B$ be the smooth normalization of B . For any subfield $l \subset k$ and for $i \in \mathbb{Z}$, we denote the kernel of the map $HH_i^l(A) \rightarrow HH_i^l(B)$ by $\overline{HH}_i^l(A, B)$.

Corollary 6.2. *Let $I \subset A$ be any given conducting ideal for the normalization $f : A \rightarrow B$ as above. Then for any subfield $l \subset k$ and any $i \geq 0$,*

(i) *The natural map*

$$\frac{HH_i^l(A/I^n)}{HH_i^l(A)} \rightarrow \frac{HH_i^l(A/I^{n_0})}{HH_i^l(A)}$$

is zero for all $n \gg n_0$.

(ii) *The natural map*

$$\frac{HH_i^l(B/I^n)}{HH_i^l(B)} \rightarrow \frac{HH_i^l(B/I^{n_0})}{HH_i^l(B)}$$

is zero for all $n \gg n_0$.

(iii) *The natural map*

$$\overline{HH}_i^l(A, B) \rightarrow HH_i^l(A/I^n)$$

is injective for all $n \gg n_0$.

Proof. This follows immediately from Proposition 5.2 and the canonical decomposition of the Hochschild homology in Theorem 6.1. \square

Corollary 6.3. *Let the notations be as in Corollary 6.2. Then for any given conducting ideal I for the normalization and for any $i, n_0 \geq 0$, the natural maps*

$$(6.1) \quad \frac{HH_i^l(B, I^n)}{HH_i^l(A, I^n)} \rightarrow \frac{HH_i^l(B, I^{n_0})}{HH_i^l(A, I^{n_0})}$$

$$(6.2) \quad \text{Ker} (HH_i^l(A, I^n) \rightarrow HH_i^l(B, I^n)) \rightarrow \text{Ker} (HH_i^l(A, I^{n_0}) \rightarrow HH_i^l(B, I^{n_0}))$$

are zero for all $n \gg n_0$.

Proof. We consider the following commutative diagram of short exact sequences coming from the long exact relative Hochschild homology sequence.

$$(6.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & \frac{HH_{i+1}^l(A/I^n)}{HH_{i+1}^l(A)} & \rightarrow & HH_i^l(A, I^n) & \rightarrow & \text{Ker} (HH_i^l(A) \rightarrow HH_i^l(A/I^n)) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{HH_{i+1}^l(B/I^n)}{HH_{i+1}^l(B)} & \rightarrow & HH_i^l(B, I^n) & \rightarrow & \text{Ker} (HH_i^l(B) \rightarrow HH_i^l(B/I^n)) \rightarrow 0 \end{array}$$

Using the naturality of the Hodge decomposition of the Hochschild homology (*loc. cit.*, Theorem 4.5.10) and smoothness of B , we have the isomorphism

$$\frac{\text{Ker}(HH_i^l(B) \rightarrow HH_i^l(B/I^n))}{\text{Ker}(HH_i^l(A) \rightarrow HH_i^l(A/I^n))} \xrightarrow{\cong} \frac{\Omega_{(B,I^n)/l}^i}{\Omega_{(A,I^n)/l}^i}.$$

Using this identification, the above diagram gives us for $n \geq 0$, an exact sequence of quotients

$$\frac{HH_{i+1}^l(B/I^n)}{HH_{i+1}^l(B)} \rightarrow \frac{HH_i^l(B, I^n)}{HH_i^l(A, I^n)} \rightarrow \frac{\Omega_{(B,I^n)/l}^i}{\Omega_{(A,I^n)/l}^i} \rightarrow 0.$$

By [15] (Lemma 4.1), one has that for any given $i, n_0 \geq 0$, the natural map $\frac{\Omega_{(B,I^n)/l}^i}{\Omega_{(A,I^n)/l}^i} \rightarrow \frac{\Omega_{(B,I^{n_0})/l}^i}{\Omega_{(A,I^{n_0})/l}^i}$ is zero for all $n \gg n_0$. Corollary 6.2 implies that for $i, n_0 \geq 0$, the natural map $\frac{HH_{i+1}^l(B/I^n)}{HH_{i+1}^l(B)} \rightarrow \frac{HH_{i+1}^l(B/I^{n_0})}{HH_{i+1}^l(B)}$ is zero for all $n \gg n_0$. Now by comparing this exact sequence for n_0 and $n \geq n_0$ and using the doubling trick, we conclude the proof of 6.1.

To prove 6.2, we observe in 6.3 that the right vertical arrow is injective for all $n \gg n_0$ by Corollary 6.2 (part (iii)). Hence we can replace the kernel of the middle vertical arrow by the kernel of the left vertical arrow in order to prove 6.2, in which case it follows from Corollary 6.2 (part (i)). \square

Corollary 6.4. *Let the notations be as in Corollary 6.2. Then for any conducting ideal I for the normalization and for any $i \in \mathbb{Z}$ and $n_0 \geq 0$, the natural maps of double relative Hochschild and cyclic homology groups*

$$(6.4) \quad HH_i^l(A, B, I^n) \rightarrow HH_i^l(A, B, I^{n_0})$$

$$(6.5) \quad HC_i^l(A, B, I^n) \rightarrow HC_i^l(A, B, I^{n_0})$$

are zero for all $n \gg n_0$.

Proof. The double relative Hochschild and cyclic homology are classically known to be zero in negative degrees. So we assume $i \geq 0$. We first prove the result for the Hochschild homology. The long exact double relative Hochschild homology sequence gives for any $i, n \geq 0$, the short exact sequence

$$0 \rightarrow \frac{HH_{i+1}^l(B, I^n)}{HH_{i+1}^l(A, I^n)} \rightarrow HH_i^l(A, B, I^n) \rightarrow \text{Ker}(HH_i^l(A, I^n) \rightarrow HH_i^l(B, I^n)) \rightarrow 0.$$

Comparing this exact sequence for n_0 and $n \geq n_0$, using Corollary 6.3 and the doubling trick, we get the desired result.

To prove the result for the cyclic homology, we use induction on $i \geq 0$.

We have $HC_i^l(A, B, I^n) = 0$ for $i < 0$ as pointed above, and hence the SBI-sequence give isomorphism $HH_0^l(A, B, I^n) \xrightarrow{\cong} HC_0^l(A, B, I^n)$. So the result holds for $i \leq 0$. Suppose the result holds for all $j \leq i - 1$ with $i \geq 1$. We have the long exact SBI-sequence

$$HH_i^l(A, B, I^n) \xrightarrow{I} HC_i^l(A, B, I^n) \xrightarrow{S} HC_{i-2}^l(A, B, I^n).$$

Comparing this exact sequence for n_0 and $n \geq n_0$, using induction on i and 6.4 (which we just proved), the doubling trick gives us the proof of 6.5. \square

Proof of Theorem 1.1: Let X be a quasi-projective variety of dimension d over a field k and let $f : \overline{X} \rightarrow X$ be the smooth normalization of X . Let $Y \hookrightarrow X$ be a given conducting subscheme for the normalization.

For $i \in \mathbb{Z}$, let $\mathcal{K}_{i,(X,\overline{X},Y)}$ denote the sheaf of double relative K -groups on X . This is a sheaf whose stalk at any point $x \in X$ is the double relative K -group $K_i(\mathcal{O}_{X,x}, \mathcal{O}_{\overline{X},x}, \mathcal{I}_{Y,x})$, where \mathcal{I}_Y is the ideal sheaf of Y . By [22], Proposition A.5 (see also [27] for more general Brown-Gersten spectral sequences), there exists a convergent spectral sequence

$${}^n E_2^{p,q} = H_{\text{Zar}}^p \left(X, \mathcal{K}_{q,(X,\overline{X},nY)} \right) \Rightarrow K_{q-p}(X, \overline{X}, nY),$$

with differential $d_r : {}^n E_r^{p,q} \rightarrow {}^n E_r^{p+r,q+r-1}$. This gives a finite filtration

$$K_i(X, \overline{X}, nY) = F_n^0 \supseteq F_n^1 \supseteq \cdots \supseteq F_n^d \supseteq F_n^{d+1} = 0$$

such that for $0 \leq j \leq d$, $F_n^j / F_n^{j+1} \cong {}^n E_\infty^{j,i+j} = {}^n E_r^{j,i+j}$, where r depends only on d and $0 \leq j \leq d$. Thus by the descending induction on j and using the doubling trick, it suffices to show that for $0 \leq j \leq d$ and $n_0 \geq 0$, the natural map ${}^n E_2^{j,i+j} \rightarrow {}^{n_0} E_2^{j,i+j}$ is zero for all $n \gg n_0$. This will be proved if we show that for $i \in \mathbb{Z}$ and $n_0 \geq 0$, the natural map of sheaves $\mathcal{K}_{i,(X,\overline{X},nY)} \rightarrow \mathcal{K}_{i,(X,\overline{X},n_0Y)}$ is zero for all $n \gg n_0$. For $i \leq 0$, these sheaves are classically known to be zero ([4]). So we assume $i \geq 1$. Since X is a k -variety, it is enough to show this when X is affine. Thus we need to show that if A is an essentially of finite type k -algebra and B is the smooth normalization of A , then for any conducting ideal I and $i, n_0 \geq 0$, the natural map $K_i(A, B, I^n) \rightarrow K_i(A, B, I^{n_0})$ is zero for all $n \gg n_0$. But this follows immediately from Corollary 6.4 (with $l = \mathbb{Q}$) and Cortinas' result ([8], Corollary 0.2) that these double relative K -groups are in fact rational vector spaces and the Chern character map $K_i(A, B, I^n) \rightarrow HC_{i-1}^{\mathbb{Q}}(A, B, I^n)$ is an isomorphism. This completes the proof of Theorem 1.1. \square

7. FORMULA FOR THE CHOW GROUP OF ZERO CYCLES

Let X be a quasi-projective variety of dimension $d \geq 2$ over a field k . We assume in this section that X is Cohen-Macaulay (all local rings are Cohen-Macaulay) and it has only isolated singularities. Note that this automatically implies that X is normal. Our aim in this section is to prove Theorem 1.2. So let $p : \tilde{X} \rightarrow X$ be a good resolution of singularities of X and let E denote the reduced exceptional divisor on \tilde{X} . Let $S \subset X$ be the singular locus of X . We give S the reduced induced subscheme structure and denote by nS , the n th infinitesimal thickening of S in X . Let $Y \hookrightarrow X$ be a closed subscheme of X such that p is the blow-up of X along Y . Then $S = Y_{\text{red}}$. Let \mathcal{I} denote the ideal sheaf for Y . Then one has

$$\tilde{X} = \text{Proj}_X(\oplus_{n \geq 0} \mathcal{I}^n) \quad \text{and} \quad E = (\text{Proj}_Y(\oplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}))_{\text{red}}.$$

Putting $\tilde{Y} = \text{Proj}_Y(\oplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1})$, we get $S \subset Y \subset nS$ and $E \subset \tilde{Y} \subset nE$ for all sufficiently large n .

Lemma 7.1. *For all $n \geq 1$, the map $F^d K_0(X, nS) \rightarrow F^d K_0(X)$ is an isomorphism. In particular, there are natural maps $F^d K_0(X) \rightarrow F^d K_0(\tilde{X}, nE)$ such that Diagram 1.1 commutes and all maps there are surjective.*

Proof. Since S is zero-dimensional, the map $F^d K_0(X, nS) \rightarrow F^d K_0(X)$ is an isomorphism by [14] (Lemma 3.1). On the other hand, there are natural maps $F^d K_0(X, nS) \rightarrow F^d K_0(\tilde{X}, nE)$ by the definition of relative K -groups (see Section 1). The isomorphism above now shows that this map factors through a map $F^d K_0(X) \rightarrow F^d K_0(\tilde{X}, nE)$. The surjectivity assertion follows from Lemmas 3.1 and 3.2 of [14]. \square

Proof of Theorem 1.2: The Northcott-Rees theory gives a minimal reduction of ideal sheaf $\mathcal{J} \subset \mathcal{I}$ of \mathcal{I} in the sense that $\mathcal{J}\mathcal{I}^n = \mathcal{I}^{n+1}$ for all sufficiently large n . Furthermore, since X is Cohen-Macaulay and S is a finite set of closed points, we can choose (cf. [29]) \mathcal{J} to be a local complete intersection ideal sheaf on X . Now we follow the proof of Theorem 1.1 of [16] to get a factorization

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow p & \searrow f & \\ & & X' \\ & \swarrow p' & \\ & & X \end{array}$$

where p' is the blow-up of X along \mathcal{J} and f is the normalization morphism. Let Y_1 denote the local complete intersection subscheme of X defined by \mathcal{J} . Since \mathcal{J} is a reduction for \mathcal{I} , we see that $Y_{1\text{red}} = Y_{\text{red}} = S$ and hence $\mathcal{I}^n \subset \mathcal{J} \subset \mathcal{I}$ for all large n . Let

$$Y' = Y \times_X X', \quad Y'_1 = Y_1 \times_X X', \quad \tilde{Y} = Y \times_X \tilde{X},$$

$$\tilde{Y}_1 = Y_1 \times_X \tilde{X}, \quad \text{and } S' = (S \times_X X')_{\text{red}}.$$

Let $Z' \subset X'$ be a conducting subscheme for the normalization map f . Put $\tilde{Z}' = Z' \times_{X'} \tilde{X}$. Then we see that $Z'_{\text{red}} \subset S'$ and $\tilde{Z}'_{\text{red}} \subset E$. In particular, given any $m > 0$, we have $mZ' \subset nS'$ and $m\tilde{Z}' \subset nE$ for all large n . Hence for a given $m > 0$, we have the following commutative diagram for all sufficiently large n with all maps surjective.

$$(7.1) \quad \begin{array}{ccccc} F^d K_0(\tilde{X}, nE) & \longrightarrow & F^d K_0(\tilde{X}, m\tilde{Z}') & \longrightarrow & F^d K_0(\tilde{X}) \\ \uparrow & & \uparrow & & \uparrow \\ F^d K_0(X', nS') & \longrightarrow & F^d K_0(X', mZ') & \longrightarrow & F^d K_0(X') \\ \uparrow & \nwarrow & \nearrow & & \\ F^d K_0(X, nS) & \longrightarrow & F^d K_0(X) & & \end{array}$$

The surjectivity of all maps follows from Lemma 3.2 of [14], and the bottom horizontal map is an isomorphism by Lemma 7.1. Now since p' is a blow-up along a local complete intersection subscheme, the map $F^d K_0(X) \rightarrow F^d K_0(X')$ is also injective by [16] (Corollary 2.5). Combining this with the surjectivity of arrows in the above diagram, we get another diagram below with all the arrows being isomorphisms.

$$\begin{array}{ccccc} F^d K_0(X', nS') & \longrightarrow & F^d K_0(X', mZ') & \longrightarrow & F^d K_0(X') \\ \uparrow & \nwarrow & \nearrow & & \\ F^d K_0(X, nS) & \longrightarrow & F^d K_0(X) & & \end{array}$$

Next we study the relation between $F^d K_0(X', mZ')$ and $F^d K_0(\tilde{X}, m\tilde{Z}')$ for fixed $m > 0$. By the long exact sequence of double relative K -theory, one has an exact sequence

$$K_0(X', \tilde{X}, mZ') \rightarrow K_0(X', mZ') \rightarrow K_0(\tilde{X}, m\tilde{Z}').$$

We compare this exact sequence for $m = 1$ and $m \gg 0$ to get a diagram

$$\begin{array}{ccccc} K_0(X', \tilde{X}, mZ') & \rightarrow & K_0(X', mZ') & \rightarrow & K_0(\tilde{X}, m\tilde{Z}') \\ \downarrow & & \downarrow & & \downarrow \\ K_0(X', \tilde{X}, Z') & \rightarrow & K_0(X', Z') & \rightarrow & K_0(\tilde{X}, \tilde{Z}') \end{array}$$

The left vertical map is zero for $m \gg 0$ by Theorem 1.1.

Put $A_m = \text{Ker} \left(F^d K_0(X', mZ') \rightarrow F^d K_0(\tilde{X}, m\tilde{Z}') \right)$. Then the above diagram gives another diagram of short exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow A_m \rightarrow F^d K_0(X', mZ') & \rightarrow & F^d K_0(\tilde{X}, m\tilde{Z}') & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow A_1 \rightarrow F^d K_0(X', Z') & \rightarrow & F^d K_0(\tilde{X}, \tilde{Z}') & \rightarrow & 0 \end{array}$$

where the left vertical map is zero for $m \gg 0$. By mapping Diagram 7.1 to a similar diagram with $m = 1$, we see that the middle vertical map above is an isomorphism.

A diagram chase shows that $F^d K_0(X', mZ') \rightarrow F^d K_0(\tilde{X}, m\tilde{Z}')$ is an isomorphism. This, together with the isomorphism $F^d K_0(X', nS') \rightarrow F^d K_0(X', mZ')$ now shows that the map $F^d K_0(X, nS) \rightarrow F^d K_0(\tilde{X}, nE)$ is an isomorphism for all large n . This gives the desired isomorphisms

$$F^d K_0(\tilde{X}, nE) \rightarrow F^d K_0(\tilde{X}, (n-1)E) \text{ and } F^d K_0(X) \rightarrow F^d K_0(\tilde{X}, nE)$$

for all large n . Finally, for X affine or projective, we get

$$CH^d(X) \cong \varprojlim_n F^d K_0(\tilde{X}, nE)$$

if k is algebraically closed. □

8. COHOMOLOGY OF MILNOR K -SHEAVES

It is by now a well known fact that algebraic cycles are closely connected to the cohomology of Quillen K -theory sheaves. This is true even for certain classes of singular varieties. However, the Quillen K -theory groups are often very difficult to compute. On the other hand, one also has the sheaves of Milnor K -theory on schemes which are relatively simpler looking objects. Our applications of Theorem 1.2 in this paper are based on the observation that for the purposes of zero cycles, the appropriate cohomology of Quillen K -theory sheaves can often be approximated by the cohomology of Milnor K -theory sheaves, which can be computed by some other means. In this section, we prove certain general reduction steps in order to use Theorem 1.2 for studying the Chow group of zero cycles on varieties with isolated singularities. In particular, we give a very precise sufficient condition for the Chow group of zero cycles on a variety X with Cohen-Macaulay isolated singularities to be isomorphic to the similar group on a resolution of singularities of X . We begin with the following result.

For any variety X over a field k , let $\mathcal{K}_{m,X}^M$ denote the sheaf of Milnor K -groups on X . This is a sheaf whose stalk at any point x of X is the Milnor K -group of the local ring $\mathcal{O}_{X,x}$. For any closed embedding $i : Y \hookrightarrow X$, let $\mathcal{K}_{m,(X,Y)}^M$ be the sheaf of relative Milnor K -groups defined so that the sequence of sheaves

$$(8.1) \quad 0 \rightarrow \mathcal{K}_{m,(X,Y)}^M \rightarrow \mathcal{K}_{m,X}^M \rightarrow i_*(\mathcal{K}_{m,Y}^M) \rightarrow 0$$

is exact. Note that the map $\mathcal{K}_{m,X}^M \rightarrow i_*(\mathcal{K}_{m,Y}^M)$ is always surjective. From now on, we shall assume our base field k to be algebraically closed unless mentioned otherwise.

Lemma 8.1. *Let X be an affine or projective variety of dimension d over k . Then there are natural isomorphisms*

$$CH^d(X) \cong F^d K_0(X) \cong H^d(X, \mathcal{K}_{d,X}^M) \cong H^d(X, \mathcal{K}_{d,X}^M).$$

Proof. The isomorphism $CH^d(X) \cong F^d K_0(X)$ was shown by Levine ([17], Corollary 2.7 and Theorem 3.2). In this case, Barvieri-Viale has shown ([2], Corollary A) that there is a natural surjection $CH^d(X) \twoheadrightarrow H^d(X, \mathcal{K}_{d,X}^M)$ with finite kernel. If X is affine, then this kernel must be zero by [17] (Theorem 2.6). If X is projective, then there is an albanese map $CH^d(X) \rightarrow H^d(X, \mathcal{K}_{d,X}^M) \rightarrow H^d(\tilde{X}, \mathcal{K}_{d,\tilde{X}}^M) \rightarrow Alb(\tilde{X})$, where \tilde{X} is a resolution of singularities of X . Now the isomorphism

$Alb(X) \cong Alb(\tilde{X})$ (since X is normal) and the Roitman torsion theorem implies that $CH^d(X) \rightarrow H^d(X, \mathcal{K}_{d,X})$ must be an isomorphism. Finally, the isomorphism $H^d(X, \mathcal{K}_{d,X}) \cong H^d(X, \mathcal{K}_{d,X}^M)$ is shown in [14] (Corollary 4.2). \square

Proposition 8.2. *Let \tilde{X} be a smooth quasi-projective variety of dimension $d+1$ over a field k . Let $E \hookrightarrow \tilde{X}$ be a strict normal crossing divisor. Then the natural cup product map*

$$H^d(E, \mathcal{K}_{m,E}^M) \otimes k^* \rightarrow H^d(E, \mathcal{K}_{m+1,E}^M)$$

is surjective for all $m \geq d$.

Proof. We prove this by induction on $d \geq 1$ and divide the proof into several cases. **Case I:** d arbitrary and E smooth.

In this case one has the Gersten resolution ([10], Proposition 4.3)

$$\begin{aligned} \mathcal{K}_{m,E}^M &\rightarrow i_* (K_m^M(k(E))) \rightarrow \bigoplus_{x \in E^{(1)}} i_* (K_{m-1}^M(k(x))) \rightarrow \cdots \\ &\rightarrow \bigoplus_{x \in E^{(d-1)}} i_* (K_{m-d+1}^M(k(x))) \rightarrow \bigoplus_{x \in E^{(d)}} i_* (K_{m-d}^M(k(x))) \rightarrow 0, \end{aligned}$$

where the first map is generically injective (in fact everywhere injective by a recent result of Kerz [13]). This resolution gives a commutative diagram

$$\begin{array}{ccc} \left(\bigoplus_{x \in E^{(d)}} i_* (K_{m-d}^M(k(x))) \right) \otimes k^* & \twoheadrightarrow & H^d(E, \mathcal{K}_{m,E}^M) \otimes k^* \\ \downarrow & & \downarrow \\ \bigoplus_{x \in E^{(d)}} i_* (K_{m+1-d}^M(k(x))) & \twoheadrightarrow & H^d(E, \mathcal{K}_{m+1,E}^M). \end{array}$$

The horizontal arrows in this diagram are surjective by the above resolution. The left vertical arrow is surjective since $k(x) = k$ for $x \in E^{(d)}$ as k is algebraically closed and the map $K_i^M(k) \otimes k_j^M(k) \rightarrow K_{i+j}^M(k)$ is surjective. A diagram chase proves the result.

Case II: $d = 1$ and E not smooth.

Let $E = E_1 \cup \cdots \cup E_r$ with $r \geq 2$ and put $E' = E_1 \cup \cdots \cup E_{r-1}$, $F_r = E' \cap E_r$. Since E is a strict normal crossing divisor on \tilde{X} , we see that F_r is a strict normal crossing divisor on E_r . Thus F_r is finite set of closed points for which the proposition is obvious. Let $\mathcal{I}_{E'}$ (resp \mathcal{I}_{E_r}) denote the ideal sheaf of E' (resp E_r) on \tilde{X} . Let \overline{E} be the closed subscheme of \tilde{X} defined by the sheaf of ideals $\mathcal{I}_{E'} \cap \mathcal{I}_{E_r}$. Then one has

$$(8.2) \quad \mathcal{K}_{m,E}^M \twoheadrightarrow \mathcal{K}_{m,\overline{E}}^M$$

and this map is generically an isomorphism. In particular, we have

$$(8.3) \quad H^d(E, \mathcal{K}_{m,E}^M) \xrightarrow{\cong} H^d(E, \mathcal{K}_{m,\overline{E}}^M) \quad \forall m.$$

Note here that E is the subscheme of \tilde{X} locally defined by the product of $\mathcal{I}_{E'}$ and \mathcal{I}_{E_r} . By [14] (Lemma 4.5), there is a short exact sequence of sheaves

$$(8.4) \quad 0 \rightarrow \mathcal{K}_{m,\bar{E}}^M \rightarrow i_* (\mathcal{K}_{m,E'}^M) \oplus i_* (\mathcal{K}_{m,E_r}^M) \rightarrow i_* (\mathcal{K}_{m,F_r}^M) \rightarrow 0.$$

Taking the cohomology exact sequences, we get a commutative diagram of exact sequences

$$(8.5) \quad \begin{array}{ccccccc} H^{d-1}(F_r, \mathcal{K}_{m,F_r}^M) \otimes k^* & \longrightarrow & H^d(E, \mathcal{K}_{m,\bar{E}}^M) \otimes k^* & \longrightarrow & H^d(E', \mathcal{K}_{m,E'}^M) \otimes k^* & \longrightarrow & 0 \\ & & \downarrow & & \oplus & & \\ & & & & H^d(E_r, \mathcal{K}_{m,E_r}^M) \otimes k^* & & \\ & & & & \downarrow & & \\ H^{d-1}(F_r, \mathcal{K}_{m+1,F_r}^M) & \longrightarrow & H^d(E, \mathcal{K}_{m+1,\bar{E}}^M) & \longrightarrow & H^d(E', \mathcal{K}_{m+1,E'}^M) & \longrightarrow & 0, \\ & & & & \oplus & & \\ & & & & H^d(E_r, \mathcal{K}_{m+1,E_r}^M) & & \end{array}$$

where the last terms in both rows are zero because $\dim(F_r) \leq d-1$. We have already observed that the left vertical arrow is surjective. The map $H^d(E_r, \mathcal{K}_{m,E_r}^M) \otimes k^* \rightarrow H^d(E_r, \mathcal{K}_{m+1,E_r}^M)$ is surjective by Case I. The map between the other summand of the right vertical arrow is surjective by induction on the number of components since $n(E') < n(E)$. We complete the proof of Case II by a diagram chase and 8.3.

Case III: $d \geq 2$. Assume by induction that the proposition holds for whenever dimension of the normal crossing divisor is $d' < d$. Let $n(E)$ denote the number of irreducible components of E . We now induct on $n(E)$. The case $n(E) = 1$ is already proved above. So assume $n(E) = n \geq 2$ and assume we have proved Case II for all normal crossing divisors E' with $n(E') < n$. Let E', E_r, F_r and \bar{E} be as in Case II. Then E' is a strict normal crossing divisor on \tilde{X} with $n(E') < n$. Moreover, E_r is smooth and F_r is a strict normal crossing divisor on E_r (if not empty) and E_r is a smooth variety of smaller dimension than \tilde{X} . So the proposition holds for E', F_r and E_r by induction and smooth case. A diagram chase again in 8.5 and 8.3 now complete the proof. \square

Proposition 8.3. *Let Z be a quasi-projective variety of dimension d over a field k and let $W = Z_{\text{red}}$. For $i \geq 0$, let $\Omega_{(Z,W)/\mathbb{Q}}^i$ denote the kernel of the map $\Omega_{Z/\mathbb{Q}}^i \rightarrow \Omega_{W/\mathbb{Q}}^i$. Then there is a natural isomorphism*

$$H^d(Z, \mathcal{K}_{d+1,(Z,W)}^M) \rightarrow H^d\left(Z, \frac{\Omega_{(Z,W)/\mathbb{Q}}^d}{\partial\left(\Omega_{(Z,W)/\mathbb{Q}}^{d-1}\right)}\right),$$

where $\partial : \Omega_{Z/\mathbb{Q}}^i \rightarrow \Omega_{Z/\mathbb{Q}}^{i+1}$ is the differential map.

Proof. Let $\phi_Z : \mathcal{K}_{m,Z}^M \rightarrow \mathcal{K}_{m,Z}$ denote the natural map from Milnor K -theory to Quillen K -theory. Since we are in characteristic 0, there exists by [26] (Theorem 12.3) a natural map

$$\psi_Z : \mathcal{K}_{m,Z} \rightarrow \mathcal{K}_{m,Z}^M$$

such that $\psi_Z \circ \phi_Z = ((-1)^{m-1}(m-1)!) \text{Id} \forall m \geq 1$. Furthermore, with respect to the γ -filtration and Adams operation on the Quillen K -theory ([18]), one has $F^{d+1}\mathcal{K}_{d+1,Z} = \mathcal{K}_{d+1,Z}^{(d+1)}$ and the map $\phi_Z : \mathcal{K}_{d+1,Z}^M \rightarrow \mathcal{K}_{d+1,Z}^{(d+1)}$ is isomorphism modulo $d!$. In particular, we get natural maps

$$(8.6) \quad \mathcal{K}_{d+1,Z}^M \xrightarrow{\phi_Z} \mathcal{K}_{d+1,Z}^{(d+1)} \xrightarrow{\psi_Z} \mathcal{K}_{d+1,Z}^M$$

which are isomorphisms modulo $d!$.

We now consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{K}_{d+1,(Z,W)}^M & \rightarrow & \mathcal{K}_{d+1,Z}^M & \rightarrow & \mathcal{K}_{d+1,W}^M \rightarrow 0 \\ & & \downarrow \phi_{ZW} & & \downarrow \phi_Z & & \downarrow \phi_W \\ 0 & \rightarrow & \overline{\mathcal{K}_{d+1,(Z,W)}^{(d+1)}} & \rightarrow & \mathcal{K}_{d+1,Z}^{(d+1)} & \rightarrow & \overline{\mathcal{K}_{d+1,W}^{(d+1)}} \rightarrow 0 \\ & & \downarrow \psi_{ZW} & & \downarrow \psi_Z & & \downarrow \psi_W \\ 0 & \rightarrow & \mathcal{K}_{d+1,(Z,W)}^M & \rightarrow & \mathcal{K}_{d+1,Z}^M & \rightarrow & \mathcal{K}_{d+1,W}^M \rightarrow 0, \end{array}$$

where the top and the bottom rows are exact by 8.1 and the group $\overline{\mathcal{K}_{d+1,(Z,W)}^{(d+1)}}$ is defined to make the middle row exact. A diagram chase gives maps ψ_{ZW} and ϕ_{ZW} which are inverses to each other modulo $d!$. Since the map $\mathcal{K}_{d+1,(Z,W)}^{(d+1)} \rightarrow \overline{\mathcal{K}_{d+1,(Z,W)}^{(d+1)}}$ is isomorphism on the smooth locus of W , we get $H^d\left(Z, \mathcal{K}_{d+1,(Z,W)}^{(d+1)}\right) \cong H^d\left(Z, \overline{\mathcal{K}_{d+1,(Z,W)}^{(d+1)}}\right)$, which in turn shows that the map

$$(8.7) \quad H^d\left(Z, \mathcal{K}_{d+1,(Z,W)}^{(d+1)}\right) \rightarrow H^d\left(Z, \mathcal{K}_{d+1,(Z,W)}^M\right)$$

is an isomorphism modulo $d!$. However, W being the reduced part of Z , $\mathcal{K}_{d+1,(Z,W)}^{(d+1)}$ is a sheaf of \mathbb{Q} -vector spaces ([5], Section 1) and $\mathcal{K}_{d+1,(Z,W)}^M$ is a sheaf of divisible groups by [14] (Sublemma 4.8). In particular, the left hand side in 8.7 is a \mathbb{Q} -vector space and the right hand side is a divisible group (since H^d is right exact on Z). This shows that the map in 8.7 must be an isomorphism.

By [5] (Theorem 1), there exists a functorial isomorphism of filtered sheaves of \mathbb{Q} -vector spaces $\mathcal{K}_{d+1,(Z,W)} \xrightarrow{\cong} \mathcal{HC}_{d,(Z,W)}$, where the filtration is given by the γ -filtration on both sides, and \mathcal{HC} denote the sheaves of cyclic homology over the base \mathbb{Q} . This gives an isomorphism $\mathcal{K}_{d+1,(Z,W)}^{(d+1)} \xrightarrow{\cong} \mathcal{HC}_{d,(Z,W)}^{(d)}$ and by 8.7, we get

$H^d \left(Z, \mathcal{K}_{d+1, (Z, W)}^M \right) \cong H^d \left(Z, \mathcal{HC}_{d, (Z, W)}^{(d)} \right)$. Thus we are reduced to showing that there is a natural isomorphism

$$(8.8) \quad H^d \left(Z, \mathcal{HC}_{d, (Z, W)}^{(d)} \right) \xrightarrow{\cong} H^d \left(Z, \frac{\Omega_{(Z, W)/\mathbb{Q}}^d}{\partial \left(\Omega_{(Z, W)/\mathbb{Q}}^{d-1} \right)} \right).$$

We have an exact sequence of sheaves

$$\mathcal{HC}_{d, (Z, W)}^{(d)} \rightarrow \mathcal{HC}_{d, Z}^{(d)} \rightarrow \mathcal{HC}_{d, W}^{(d)} \rightarrow 0,$$

where the last term is zero since

$$\mathcal{HC}_{d, Z}^{(d)} \cong \frac{\Omega_Z^d / \mathbb{Q}}{\partial \left(\Omega_Z^{d-1} / \mathbb{Q} \right)} \twoheadrightarrow \frac{\Omega_W^d / \mathbb{Q}}{\partial \left(\Omega_W^{d-1} / \mathbb{Q} \right)} \cong \mathcal{HC}_{d, W}^{(d)}$$

by [20] (Theorem 4.6.8). Furthermore, since $\mathcal{O}_Z \twoheadrightarrow \mathcal{O}_W$ is locally split on the smooth locus of W , we see that the first map in the above exact sequence is injective on the smooth locus of W . By the same reason, there is a natural surjection

$$\frac{\Omega_{(Z, W)/\mathbb{Q}}^d}{\partial \left(\Omega_{(Z, W)/\mathbb{Q}}^{d-1} \right)} \twoheadrightarrow \text{Ker} \left(\frac{\Omega_Z^d / \mathbb{Q}}{\partial \left(\Omega_Z^{d-1} / \mathbb{Q} \right)} \twoheadrightarrow \frac{\Omega_W^d / \mathbb{Q}}{\partial \left(\Omega_W^{d-1} / \mathbb{Q} \right)} \right),$$

which is an isomorphism on the smooth locus of W . In particular, we get surjective maps

$$\mathcal{HC}_{d, (Z, W)}^{(d)} \twoheadrightarrow \text{Ker} \left(\frac{\Omega_Z^d / \mathbb{Q}}{\partial \left(\Omega_Z^{d-1} / \mathbb{Q} \right)} \twoheadrightarrow \frac{\Omega_W^d / \mathbb{Q}}{\partial \left(\Omega_W^{d-1} / \mathbb{Q} \right)} \right) \leftarrow \frac{\Omega_{(Z, W)/\mathbb{Q}}^d}{\partial \left(\Omega_{(Z, W)/\mathbb{Q}}^{d-1} \right)}$$

which are isomorphisms on the smooth locus of W , and hence they induce isomorphisms on the top cohomology H^d . This proves 8.8 and hence the proposition. \square

Corollary 8.4. *Let X be either an affine or a projective variety of dimension $d \geq 2$ over a field k . Assume that X is Cohen-Macaulay and has only isolated singularities. Let $p : \tilde{X} \rightarrow X$ be a good resolution of singularities of X such that the exceptional divisor E is a strict normal crossings divisor (such a resolution always exists). Then the map $CH^d(X) \rightarrow CH^d(\tilde{X})$ is an isomorphism if the following two conditions hold.*

- (i) *The map $H^{d-1} \left(\tilde{X}, \mathcal{K}_{d-1, \tilde{X}} \right) \otimes k^* \rightarrow H^{d-1} (E, \mathcal{K}_{d-1, E}) \otimes k^*$ is surjective.*
- (ii) *$H^{d-1} \left(nE, \frac{\Omega_{(nE, E)/\mathbb{Q}}^{d-1}}{\partial \left(\Omega_{(nE, E)/\mathbb{Q}}^{d-2} \right)} \right) = 0$ for all $n \geq 1$.*

Proof. By Theorem 1.1, we need to show that the map $F^d K_0(\tilde{X}, nE) \rightarrow F^d K_0(\tilde{X})$ is an isomorphism for all large n . By [14] (Proposition 4.3), this further reduces to showing that the map $H^d \left(\tilde{X}, \mathcal{K}_{d, (\tilde{X}, nE)}^M \right) \rightarrow H^d \left(\tilde{X}, \mathcal{K}_{d, \tilde{X}}^M \right)$ is an isomorphism for

all $n \geq 1$. Considering the long exact cohomology sequence corresponding to 8.1, $H^{d-1}(\tilde{X}, \mathcal{K}_{d,\tilde{X}}^M) \rightarrow H^{d-1}(nE, \mathcal{K}_{d,nE}^M) \rightarrow H^d(\tilde{X}, \mathcal{K}_{d,(\tilde{X},nE)}^M) \rightarrow H^d(\tilde{X}, \mathcal{K}_{d,\tilde{X}}^M) \rightarrow 0$, we need to show that the first map on the left is surjective for all n .

First we consider the case $n = 1$. We have seen in the proof of Proposition 8.3 that there is a natural map $\psi_E : \mathcal{K}_{d-1,E} \rightarrow \mathcal{K}_{d-1,E}^M$ whose cokernel is of fixed exponent $(d-2)!$. In particular, the cokernel of the map $H^{d-1}(E, \mathcal{K}_{d-1,E}) \rightarrow H^{d-1}(E, \mathcal{K}_{d-1,E}^M)$ is of finite exponent (since H^{d-1} is right exact on E). However, as k^* is a divisible group ($k = \bar{k}$), we must have

$$(8.9) \quad H^{d-1}(E, \mathcal{K}_{d-1,E}) \otimes k^* \twoheadrightarrow H^{d-1}(E, \mathcal{K}_{d-1,E}^M) \otimes k^*.$$

Next we have a commutative diagram

$$\begin{array}{ccc} H^{d-1}(\tilde{X}, \mathcal{K}_{d-1,\tilde{X}}^M) \otimes k^* & \longrightarrow & H^{d-1}(\tilde{X}, \mathcal{K}_{d,\tilde{X}}^M) \\ \downarrow & & \downarrow \\ H^{d-1}(E, \mathcal{K}_{d-1,E}^M) \otimes k^* & \longrightarrow & H^{d-1}(E, \mathcal{K}_{d,E}^M). \end{array}$$

The first condition of the corollary and 8.9 together imply that the left vertical arrow is surjective. The bottom horizontal arrow is surjective by Proposition 8.2. A diagram chase shows that the right vertical arrow is also surjective. This finishes the case $n = 1$.

Now assume $n \geq 2$. Then 8.1 for the pair $E \hookrightarrow nE$ gives an exact sequence

$$H^{d-1}(nE, \mathcal{K}_{d,(nE,E)}^M) \rightarrow H^{d-1}(nE, \mathcal{K}_{d,nE}^M) \rightarrow H^{d-1}(E, \mathcal{K}_{d,E}^M) \rightarrow 0.$$

The second condition of the corollary and Proposition 8.3 together imply that the group on the left is zero. This completes the proof. \square

9. CHOW GROUP OF AFFINE CONES

Let $Y \hookrightarrow \mathbb{P}_k^N$ be a smooth projective variety of dimension d over a field k . Let $X = C(Y)$ be the affine cone over Y and let $\overline{X} \hookrightarrow \mathbb{P}_k^{N+1}$ be the projective cone over Y . Let $P \in X \hookrightarrow \overline{X}$ denote the vertex of the cone. We assume that X is Cohen-Macaulay. Since P is the only singular point of both X and \overline{X} , we see that \overline{X} is also Cohen-Macaulay. Let $p : \tilde{X} \rightarrow X$ be the blow-up of X along the vertex P and let $E = p^{-1}(P)$ be the exceptional divisor for the blow-up. This situation gives rise to the following commutative diagram.

$$(9.1) \quad \begin{array}{ccccc} E \hookrightarrow \tilde{X} & \xrightarrow{p} & X & & \\ & \searrow & \downarrow & \searrow & \\ & & Z & \xrightarrow{\bar{p}} & \overline{X} \\ & \searrow \pi & \downarrow \bar{\pi} & & \\ & & Y & & \end{array}$$

The map \bar{p} is the blow-up of \bar{X} along P . The map π is the \mathbb{A}^1 -bundle over Y associated to the ample line bundle $\mathcal{O}_Y(1)$ with the zero-section E and $\bar{\pi}$ is the \mathbb{P}^1 -bundle over Y associated to the vector bundle $\mathcal{O}_Y \oplus \mathcal{O}_Y(1)$. In particular, \tilde{X} (resp Z) is a good resolution of singularities of X (resp \bar{X}) such that the exceptional divisor $E \hookrightarrow \tilde{X} \hookrightarrow Z$ is smooth and the inclusion of E in \tilde{X} and Z has sections given by the maps π and $\bar{\pi}$. Our aim in this section is to prove Theorem 1.5 for which we need the following preliminary results.

Lemma 9.1. *Let \tilde{X} be a smooth quasi-projective variety of dimension $d + 1$ over a field k and let $E \hookrightarrow \tilde{X}$ be a smooth divisor such that*

$$H^d \left(E, \Omega_{E/k}^i \otimes_E \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \right) = 0 \text{ for } i \geq 0 \text{ and } n \geq 1,$$

where \mathcal{I} is the ideal sheaf of E on \tilde{X} . Then

$$H^d (nE, \Omega_{(nE, E)/k}^i) = 0 \text{ for } i \geq 0 \text{ and } n \geq 1.$$

Proof. We first claim that

$$(9.2) \quad H^d \left(nE, \Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n} \right) = 0 \quad \forall i \geq 0, n \geq 1.$$

To prove the claim, we consider for $i \geq 0$ and $n \geq 1$ the compatible maps

$$0 = \Omega_{nE/k}^i \otimes \frac{\mathcal{I}^n}{\mathcal{I}^n} \rightarrow \Omega_{nE/k}^i \otimes \frac{\mathcal{I}^{n-1}}{\mathcal{I}^n} \rightarrow \cdots \rightarrow \Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n}$$

and put $F_j^i = \text{Image} \left(\Omega_{nE/k}^i \otimes \frac{\mathcal{I}^j}{\mathcal{I}^n} \rightarrow \Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n} \right)$ for $1 \leq j \leq n$. This gives a finite filtration $\{F_j^i\}_{1 \leq j \leq n}$ of $\Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n}$ such that

$$(9.3) \quad \Omega_{nE/k}^i \otimes_{nE} \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}} \twoheadrightarrow \frac{F_j^i}{F_{j+1}^i} \quad \forall 1 \leq j \leq n.$$

Since H^d is right exact on nE , this filtration also gives exact sequence

$$H^d (nE, F_{j+1}^n) \rightarrow H^d (nE, F_j^n) \rightarrow H^d \left(nE, \frac{F_j^i}{F_{j+1}^i} \right) \rightarrow 0$$

for $1 \leq j \leq n$. Thus by a descending induction on j and by 9.3, it suffices to show that

$$(9.4) \quad H^d \left(nE, \Omega_{nE/k}^i \otimes_{nE} \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}} \right) = 0 \quad \forall 1 \leq j \leq n$$

in order to prove the claim. If $n = 1$ or $i = 0$, this is our assumption. So assume $n \geq 2$ and $i \geq 1$. Then we have

$$\Omega_{nE/k}^i \otimes_{nE} \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}} \cong \Omega_{nE/k}^i \otimes_{nE} \left(\mathcal{O}_E \otimes_E \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}} \right) \cong \overline{\Omega_{nE/k}^i} \otimes_E \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}},$$

where $\overline{\Omega_{nE/k}^i} = \Omega_{nE/k}^i \otimes_{nE} \mathcal{O}_E$. Next we have short exact sequence

$$(9.5) \quad 0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \rightarrow \overline{\Omega_{nE/k}^1} \rightarrow \Omega_{E/k}^1 \rightarrow 0,$$

where the first term is zero because the long exact sequence of André-Quillen homology would tell us that this term would otherwise be $D_1(E/k)$ which vanishes as E is smooth. Since E is a smooth divisor on a smooth variety \tilde{X} , $\mathcal{L} = \frac{\mathcal{I}}{\mathcal{I}^2}$ is a line bundle on E and $\Omega_{E/k}^1$ is clearly a vector bundle on E . This implies that $\overline{\Omega_{nE/k}^1}$ is also a vector bundle on E and for $i \geq 1$, the obvious filtration of $\wedge_E^i \left(\overline{\Omega_{nE/k}^1} \right) = \overline{\Omega_{nE/k}^i}$ in terms of the tensor product of the exterior powers of \mathcal{L} and $\Omega_{E/k}^1$ gives us an exact sequence

$$(9.6) \quad 0 \rightarrow \Omega_{E/k}^{i-1} \otimes_E \mathcal{L} \rightarrow \overline{\Omega_{nE/k}^i} \rightarrow \Omega_{E/k}^i \rightarrow 0.$$

Tensoring this with \mathcal{L}^j and observing that $\mathcal{L}^j = \frac{\mathcal{I}^j}{\mathcal{I}^{j+1}}$ for $j \geq 1$, we get exact sequence

$$(9.7) \quad 0 \rightarrow \Omega_{E/k}^{i-1} \otimes_E \mathcal{L}^{j+1} \rightarrow \overline{\Omega_{nE/k}^i} \otimes_E \mathcal{L}^j \rightarrow \Omega_{E/k}^i \otimes_E \mathcal{L}^j \rightarrow 0.$$

Using the right exactness of H^d as before and our assumption, we get 9.4 and hence 9.2.

Finally, to prove the lemma, we consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{(nE,E)/k}^i & \longrightarrow & \Omega_{nE/k}^i & \longrightarrow & \Omega_{E/k}^i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Omega_{E/k}^{i-1} \otimes_E \mathcal{L} & \longrightarrow & \overline{\Omega_{nE/k}^i} & \longrightarrow & \Omega_{E/k}^i \longrightarrow 0, \end{array}$$

where the bottom row is the exact sequence of 9.6. Observing that the kernel of the middle vertical arrow is a quotient of $\Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n}$, a diagram chase above gives an exact sequence

$$\Omega_{nE/k}^i \otimes \frac{\mathcal{I}}{\mathcal{I}^n} \rightarrow \Omega_{(nE,E)/k}^i \rightarrow \Omega_{E/k}^{i-1} \otimes_E \mathcal{L} \rightarrow 0.$$

Now the lemma follows from the right exactness of H^d on nE , the above claim 9.2 and our assumption. \square

Lemma 9.2. *Under the conditions of Lemma 9.1, the natural map*

$$H^d \left(nE, \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_{nE} \right) \rightarrow H^d \left(E, \Omega_{E/k}^i \right)$$

is an isomorphism for $i \geq 0$ and $n \geq 1$.

Proof. The proof follows from the arguments very similar to that in the proof of Lemma 9.1. We give a sketch. First we show this when $n = 1$. In this case, the argument used in the proof of the exact sequence 9.6 also gives the exact sequence

$$0 \rightarrow \Omega_{E/k}^{i-1} \otimes_E \mathcal{L} \rightarrow \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_E \rightarrow \Omega_{E/k}^i \rightarrow 0.$$

Taking the exact sequence of H^d on E , our hypothesis now proves the case $n = 1$.

For $n \geq 2$, we can use the $n = 1$ case to reduce the proof of the lemma to showing that for $n \geq 2$ the natural map

$$(9.8) \quad H^d \left(nE, \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_{nE} \right) \rightarrow H^d \left(E, \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_E \right)$$

is an isomorphism. We have the short exact sequence of sheaves on \tilde{X}

$$0 \rightarrow \mathcal{I}/\mathcal{I}^n \rightarrow \mathcal{O}_{nE} \rightarrow \mathcal{O}_E \rightarrow 0,$$

which in turn gives the exact sequence

$$0 \rightarrow \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{I}/\mathcal{I}^n \rightarrow \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_{nE} \rightarrow \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{O}_E \rightarrow 0,$$

as $\Omega_{\tilde{X}/k}^i$ is a vector bundle on \tilde{X} . Taking the exact sequence of H^d on nE , we only need to show that $H^d \left(nE, \Omega_{\tilde{X}/k}^i \otimes_{\tilde{X}} \mathcal{I}/\mathcal{I}^n \right) = 0$ for $n \geq 2$. But this is proved exactly in the same way as we proved 9.2. \square

Lemma 9.3. *Under the conditions of Lemma 9.1, one has*

$$H^d \left(nE, \Omega_{(nE, E)/\mathbb{Q}}^i \right) = 0 \text{ for } i \geq 0 \text{ and } n \geq 1.$$

Proof. The spectral sequence of 5.1 gives a finite filtration

$$\Omega_{nE/\mathbb{Q}}^i = F_n^0 \supseteq F_n^1 \supseteq \cdots \supseteq F_n^i \supseteq F_n^{i+1} = 0$$

of $\Omega_{nE/\mathbb{Q}}^i$ and a morphism of filtered modules $\Omega_{nE/\mathbb{Q}}^i \twoheadrightarrow \Omega_{(n-1)E/\mathbb{Q}}^i$ such that $\Omega_{k/\mathbb{Q}}^i \otimes_k \Omega_{nE/k}^{i-j} \twoheadrightarrow F_n^j / F_n^{j+1}$ for $0 \leq j \leq i$. We also see immediately from the spectral sequence 5.1 and by a descending induction on j that $F_n^j \twoheadrightarrow F_1^j$ for $0 \leq j \leq i$ and $n \geq 1$. In particular, $\Omega_{(nE, E)/\mathbb{Q}}^i = \text{Ker} \left(\Omega_{nE/\mathbb{Q}}^i \twoheadrightarrow \Omega_{E/\mathbb{Q}}^i \right)$ has a filtration

$$\Omega_{(nE, E)/\mathbb{Q}}^i = F_{(n,1)}^0 \supseteq F_{(n,1)}^1 \supseteq \cdots \supseteq F_{(n,1)}^i \supseteq F_{(n,1)}^{i+1} = 0$$

with $F_{(n,1)}^j = \text{Ker} \left(F_n^j \twoheadrightarrow F_1^j \right)$ and

$$F_{(n,1)}^j / F_{(n,1)}^{j+1} = \text{Ker} \left(F_n^j / F_n^{j+1} \twoheadrightarrow F_1^j / F_1^{j+1} \right) \text{ for } 0 \leq j \leq i.$$

On the other hand, the spectral sequence 5.1 also shows that $\Omega_{k/\mathbb{Q}}^i \otimes_k \Omega_{E/k}^{i-j} \cong F_1^j / F_1^{j+1}$ for $0 \leq j \leq i$ as E is smooth, and this gives

$$\Omega_{k/\mathbb{Q}}^i \otimes_k \Omega_{(nE, E)/k}^{i-j} \twoheadrightarrow F_{(n,1)}^j / F_{(n,1)}^{j+1}.$$

This in turn gives a surjection

$$(9.9) \quad \Omega_{k/\mathbb{Q}}^i \otimes_k H^d \left(nE, \Omega_{(nE, E)/k}^{i-j} \right) \twoheadrightarrow H^d \left(nE, F_{(n,1)}^j / F_{(n,1)}^{j+1} \right) \text{ for } 0 \leq j \leq i \text{ and } n \geq 1.$$

Now the lemma follows by using the exact sequence

$$H^d \left(nE, F_{(n,1)}^{j+1} \right) \rightarrow H^d \left(nE, F_{(n,1)}^j \right) \rightarrow H^d \left(nE, F_{(n,1)}^j / F_{(n,1)}^{j+1} \right) \rightarrow 0,$$

9.9, Lemma 9.1 and a descending induction on j . \square

Proof of Theorem 1.5: We first show at once that (i) implies (iv) and (v). We follow the diagram 9.1 and the notations as in the beginning of this section. Since \tilde{X} (resp Z) is an \mathbb{A}^1 -bundle (resp a \mathbb{P}^1 -bundle) over Y , it is easy to show that $CH^{d+1}(\tilde{X}) = 0$ and $CH^{d+1}(Z) \cong CH^d(Y)$. Thus it suffices to show that the natural maps

$$(9.10) \quad CH^{d+1}(X) \rightarrow CH^{d+1}(\tilde{X}) \text{ and } CH^{d+1}(\overline{X}) \rightarrow CH^{d+1}(Z)$$

are isomorphisms. We prove the first isomorphism. The proof of the second isomorphism is exactly the same, once we observe that $E \hookrightarrow \tilde{X} \hookrightarrow Z$ is the exceptional divisor for both p and \bar{p} . We only need to verify the two conditions of Corollary 8.4. The first condition is obvious since the inclusion $E \hookrightarrow \tilde{X}$ has a section. To prove the second condition, it is enough to show that $H^d \left(nE, \Omega_{(nE, E)/\mathbb{Q}}^i \right) = 0$, since $\dim(nE) = d$. By Lemma 9.3, this further reduces to showing that

$$(9.11) \quad H^d \left(E, \Omega_{E/k}^i \otimes_E \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \right) = 0 \text{ for } i \geq 0 \text{ and } n \geq 1.$$

Let $T \subset Y$ be a hyperplane section of Y for the given embedding $Y \hookrightarrow \mathbb{P}_k^N$. Then for $n \geq 1$, one has as short exact sequence

$$0 \rightarrow \mathcal{O}_Y(n-1) \rightarrow \mathcal{O}_Y(n) \rightarrow \mathcal{O}_T(n) \rightarrow 0.$$

Since T is $(d-1)$ -dimensional, our assumption now immediately implies that $H^d(Y, \mathcal{O}_Y(n)) = 0 \forall n \geq 1$. However, since π is an \mathbb{A}^1 -bundle over Y associated to the line bundle $\mathcal{O}_Y(1)$ with a section E , we have $\mathcal{I}^n / \mathcal{I}^{(n+1)} \cong \mathcal{O}_Y(n)$ for all $n \geq 1$. Hence we get

$$(9.12) \quad H^d(E, \mathcal{I}^n / \mathcal{I}^{(n+1)}) = 0 \forall n \geq 1.$$

This proves 9.11 for $i = 0$. For $i \geq 1$, we have $H^d \left(E, \Omega_{E/k}^i \otimes_E \mathcal{I}^n / \mathcal{I}^{(n+1)} \right) \cong H^d \left(Y, \Omega_{Y/k}^i(n) \right)$ under the isomorphism $E \xrightarrow{\cong} Y$. But this last group is zero for $n \geq 1$ and $i \geq 1$ by the Akizuki-Nakano vanishing theorem (cf. [11], Theorem 1.3). This proves 9.11 and hence (i) implies (iv) and (v).

Now we assume that k is a universal domain. In this case, the implication (iv) \Rightarrow (i) was shown by Srinivas in [25] (Corollary 2). Before we prove the other implications, we first observe that as \overline{X} has only isolated singularities, the map

$H^{d+1}(\overline{X}, \Omega_{\overline{X}/k}^i) \rightarrow H^{d+1}(\overline{X}, \overline{p}_* (\Omega_{Z/k}^i))$ is an isomorphism for $i \geq 0$. Now the Leray spectral sequence gives us for $i \geq 0$, an exact sequence (9.13)

$$H^d(Z, \Omega_{Z/k}^i) \rightarrow \varprojlim_n H^d(nE, \Omega_{Z/k}^i \otimes_Z \mathcal{O}_{nE}) \rightarrow H^{d+1}(\overline{X}, \Omega_{\overline{X}/k}^i) \rightarrow H^{d+1}(Z, \Omega_{Z/k}^i) \rightarrow 0.$$

Furthermore, since $\overline{\pi}$ is a \mathbb{P}^1 -bundle, we can use the Leray spectral sequence again to get

$$(9.14) \quad H^i(Z, \mathcal{O}_Z) \cong H^i(Y, \mathcal{O}_Y) \cong H^i(E, \mathcal{O}_E) \quad \forall i \geq 0 \text{ and}$$

$$H^i(Z, \Omega_{Z/k}^r) \twoheadrightarrow H^i(E, \Omega_{E/k}^r) \quad \forall i \geq 0 \text{ } r \geq 1.$$

Proof of (i) \Leftrightarrow (ii). Suppose $H^{d+1}(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$. Then (i) follows from 9.13 for $i = 0$ and 9.14, once we observe that $E \hookrightarrow nE$ has a section for all $n \geq 1$. If (i) holds, then we have already seen in 9.2 that $H^d(nE, \mathcal{I}/\mathcal{I}^n) = 0$ for $n \geq 1$ and hence $H^d(nE, \mathcal{O}_{nE}) \cong H^d(E, \mathcal{O}_E)$ for all $n \geq 1$. Now (ii) follows from 9.13 for $i = 0$ and 9.14.

Proof of (ii) \Leftrightarrow (iii). We only need to show that (ii) \Rightarrow (iii). Since $H^{d+1}(W, \Omega_{W/k}^i)$ are birational invariants of W , we can replace W by Z everywhere below. It suffices then to show that $H^{d+1}(\overline{X}, \Omega_{\overline{X}/k}^i) \cong H^{d+1}(Z, \Omega_{Z/k}^i)$ for $i \geq 1$. By 9.13 and 9.14, this reduces to showing that $H^d(nE, \Omega_{nE/k}^i \otimes_{nE} \mathcal{I}/\mathcal{I}^n) = 0$ for $n \geq 1$. But this follows from the Akizuki-Nakano vanishing theorem and the claim 9.2 in the proof of Lemma 9.1.

Proof of (iii) \Leftrightarrow (iv). We have already shown that (iii) \Leftrightarrow (i) and (i) \Leftrightarrow (iv). So we get (iii) \Leftrightarrow (iv).

Proof of (iv) \Leftrightarrow (v). We have seen before that (iv) \Leftrightarrow (i) \Rightarrow (v). So we are only left to show that (v) \Rightarrow (iv).

For this, it suffices to show that $\text{Ker}(CH^{d+1}(\overline{X}) \rightarrow CH^{d+1}(Z)) \twoheadrightarrow CH^{d+1}(X)$. Let $H \hookrightarrow \overline{X}$ be the hyperplane at infinity, i.e., H is the complement of X in \overline{X} and is isomorphic to Y . Moreover, the inclusion $H \hookrightarrow \overline{X}$ factors through the inclusion $H \xrightarrow{j} Z$ and the natural map $CH^d(H) \xrightarrow{j_*} CH^{d+1}(Z)$ is an isomorphism. Now we have the following commutative diagram

$$\begin{array}{ccccccc} & & F & \twoheadrightarrow & CH^d(H) & & \\ & & \downarrow & & \downarrow j_* & & \\ 0 & \rightarrow & \text{Ker}(\overline{p}^*) & \rightarrow & CH^{d+1}(\overline{X}) & \rightarrow & CH^{d+1}(Z) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Ker}(p^*) & \rightarrow & CH^{d+1}(X) & \rightarrow & CH^{d+1}(\tilde{X}) \rightarrow 0, \end{array}$$

where F is the free abelian group on the closed points of H . Note that as H is contained in the smooth locus of \overline{X} , the map $F \rightarrow H^{d+1}(\overline{X})$ is well defined and the composite $F \rightarrow H^{d+1}(\overline{X}) \rightarrow CH^{d+1}(X)$ is clearly zero. We already know that $CH^{d+1}(\tilde{X}) = 0$. Since j_* is an isomorphism, a diagram chase shows that $\text{Ker}(\bar{p}^*) \rightarrow CH^{d+1}(X)$. This completes the proof of Theorem 1.5. \square

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